Chapter 1

Root Concepts of the Standard Model

The standard model of particle physics accurately describes a vast range of phenomena using a small number of parameters. Much of the power of the standard model arises from the fact that it embodies a few deep physical and mathematical concepts which are difficult, but not quite impossible, to make consistent with one another. These principles include, first of all, the general principles of special relativity and quantum mechanics. Those lead us to local quantum field theory. To these general principles, the standard model adds one more specific ingredient: local, or gauge, symmetry. Gauge symmetry is a vast generalization of the principle of electric charge conservation.

Formulating equations that embody these concepts is only half the job, because it is not at all straightforward to see that the resulting equations describe Nature. Specially, apart from the special case of quantum electrodynamics, which forms only a small sub-theory of the standard model, local gauge symmetry is not manifest in the superficial appearance of phenomena. In applying the basic highly symmetrical equations to describe observed reality, which appears much less symmetric, two profound dynamical effects must be taken into considerations.

The first of these dynamical effects is the spontaneous breaking of local gauge symmetry. It is a form of super-conductivity, but operating in (what we perceive as) empty space. In conventional super-conductivity, roughly speaking, the electrons in a metal, which normally behave as a gas of independent particles, condense into a liquid of overlapping Cooper pairs. The dynamical response of this liquid screens the electromagnetic interactions, and renders magnetic fields short-ranged. That is the essence of the Meissner effect. In the context of the standard model, the role of Cooper pairs is played by a new form of matter, the so-called Higgs field, or more accurately the Higgs multiplet of fields. The Higgs multiplet is a theoretical

construct that was invented specially to fulfill this mission. According to the theory, a condensate of Higgs particles fills empty space, and its dynamical response screens the weak interactions, rendering them short-ranged.

There is weighty indirect evidence for the existence of the Higgs multiplet, but at present direct observation of its quanta, the Higgs particles, remains a major unmet challenge, as we shall discuss in detail later.

The second of these dynamical effects is confinement of quarks. Confinement should be considered together with the closely related property of asymptotic freedom, which is in a sense its inverse. Confinement and asymptotic freedom occur in the sector of the standard model dealing with the strong interaction, quantum chromodynamics or QCD. In this context, local gauge symmetry takes a most peculiar form. The fundamental building-blocks of the theory – quarks and gluons – transform non-trivially under an $SU(3)$ local gauge symmetry. Indeed, the so-called color charges of these particles, which specify their transformation properties, are entirely responsible for their QCD interactions. But the physical particles in QCD are all singlets, which do not transform under the gauge symmetry. They are formed out of combinations of quarks, anti-quarks, and gluons in which the color charges all cancel.

Although the colored building blocks are real and tangible, and reveal their existence quite directly to suitable probes, they cannot be separated out and examined individually. Attempts to pull them apart call ever-growing forces into play, and are inevitably frustrated. This is confinement. Yet when the color charges are close together, or when we consider processes that involve large changes in energy and momentum, the forces are feeble an d the radiation is rare. This is asymptotic freedom. These unusual behaviors – fundamental forces that grow with distance, radiators that become quiescent as they are shaken violently – were once thought to be paradoxical or even problematic. They are now understood to be a general feature of many model theories with local gauge symmetry. They are deeply related, respectively, to the most basic formulation of local gauge symmetry, and to the ultimate consistency of quantum field theory.

In QCD itself, asymptotic freedom is born out of a interplay among our three basic concepts of relativity, quantum mechanics, and gauge symmetry. All three play crucial roles, as we shall see in detail later.

Chapter 2

Abelian Gauge Symmetry

As I have already mentioned, local gauge symmetry is the major new principle, beyond the generic implementation of special relativity and quantum mechanics in quantum field theory, which arises in formulating the standard model. It is, however, a rather abstract concept, and one whose relevance to the description of Nature emerges only after considerable analysis. One could simply introduce local gauge symmetry as a mathematical postulate, but that would make its significance seem obscure and its discovery miraculous. Since our goal is to construct a rational understanding of Nature, it is appropriate to proceed differently. Accordingly I will work up to the general abstract concept through enlightening special cases, and say a few words about the historical roots o f the ideas.

Physicists first encountered gauge symmetry as an esoteric feature of the Maxwell equations of electrodynamics. It was found that these equations took on a simpler mathematical form, and could be more easily solved, if one introduced potentials. The electric and magnetic fields, which form the observable content of the theory, are constructed as certain combinations of the time and space derivatives of the potentials. The correspondence between fields and potentials is not unique, in that different potentials can correspond to the same fields. Gauge symmetry is a family of functional transformations among the potentials that leaves the field strength unchanged. The charge and current distributions, which provide the source terms in the Maxwell equations, are under gauge transformations.

In the general, classical continuum form the Maxwell equations the physical meaning of gauge symmetry is quite obscure. It becomes somewhat clearer if we work in a manifestly (special) relativistic formulations.

The only simple relativistic invariant associated with a structure-less point particle is the interval of proper time $d\tau$, where

$$d\tau^2 = dt^2 - d\vec{x}^2 \tag{2.1}$$

This motivates the action

$$S_{free} = -m \int d\tau = -\int dt\ m\sqrt{1-\vec{v}^2} \equiv \int dt L_{free} \tag{2.2}$$

with $L_{free} = -m\sqrt{1-\vec{v}^2}$, where v is of course the velocity. This leads to the momentum

$$p^j_{free} = \frac{\partial L_{free}}{\partial \dot{x}^j} = m\frac{v^j}{\sqrt{1-v^2}} \tag{2.3}$$

which has the expected form for a free particle, and identifies m as the mass. (Note that the spatial momentum $p^j = -p_j$.)

To couple this particle to a "vector" field $A_\mu(\vec{x},t)$ – whose components we shall from here on call potentials – we add to the action a term

$$S_{int.} = -q \int A_\mu dx^\mu \tag{2.4}$$

corresponding to the Lagrangian

$$L_{int.} = -qA_0 + qA^j \frac{dx^j}{dt}. \tag{2.5}$$

The momentum is now, with $L \equiv L_{free} + L_{int.}$,

$$p^j = \frac{\partial L}{\partial \dot{x}^j} = m\frac{v^j}{\sqrt{1-v^2}} + qA^j \tag{2.6}$$

and

$$\frac{dp^j}{dt} = \frac{d}{dt}\left(\frac{mv^j}{\sqrt{1-\vec{v}^2}}\right) + q\frac{\partial A^j}{\partial x^k}\frac{dx^j}{dt} + q\frac{\partial A^j}{\partial t}. \tag{2.7}$$

Since

$$\frac{\partial L}{\partial x^j} = -q\frac{\partial A_0}{\partial x^j} + q\frac{\partial A^k}{\partial x^j}\frac{dx^k}{dt} \tag{2.8}$$

the equation of motion $\frac{dp^j}{dt} = \frac{\partial L}{\partial x^j}$ becomes

$$\frac{d}{dt}\left(\frac{mv^j}{\sqrt{1-\vec{v}^2}}\right) = q\left(-\frac{\partial A_0}{\partial x^j} - \frac{\partial A^j}{\partial t}\right) + \left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A^j}{\partial x^k}\right)\frac{dx^k}{dt} \tag{2.9}$$

Separating out the terms on the right hand side which contain the velocity, and adopting vector notation, we can write the equation of motion as

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{2.10}$$

with

$$E^j \equiv -\frac{\partial A_0}{\partial x^j} - \frac{\partial A^j}{\partial t} \tag{2.11}$$

and

$$B^j \equiv \epsilon^{jlm}\frac{\partial A^l}{\partial x^m} \tag{2.12}$$

Identifying $\vec{E}$ and $\vec{B}$ as electric and magnetic field strengths, we thereby arrive at the Lorentz force law for a particle of mass m, charge q.

Note that with this identification two of the Maxwell equations, viz.

$$\frac{\partial B^j}{\partial x^j} = 0 \tag{2.13}$$

$$\epsilon^{jlm}\frac{\partial E^l}{\partial x^m} + \frac{\partial B^j}{\partial t} = 0 \tag{2.14}$$

are satisfied identically. Historically, this was one main motivation for introducing the potentials A_0 and A^j. Another was their appearance in the expressions for the conserved energy and momentum, respectively, of charged particles. Here we have seen how naturally potentials arise in the quasi-geometric framework of relativistic point particle kinematics.

The transformation

$$A'_\mu = A_\mu + \partial_\mu \chi \tag{2.15}$$

where χ is a smooth function of space and time that vanishes at infinity, manifestly leaves the action $S_{int.}$ invariant, as long as the world-line has no ends. Indeed, one can simply integrate by parts in $\int dx^\mu \partial_\mu \chi$. Equation 2.15 is the transformation that defines local gauge symmetry. Since local gauge symmetry leaves the equation of motion unchanged, it must leave $\vec{E}$ and $\vec{B}$ unchanged, as of course one can verify directly.

Clearly, the requirement that the world-line of a charged particle should have no ends is closely related to the conservation of charge. More generally, the necessary and sufficient condition for an action

$$S_{int.} = -\int d^3x dt\ A_\mu j^\mu \tag{2.16}$$

to be invariant under local gauge symmetry (i.e, with the gauge function χ vanishing at infinity) i s the "continuity equation"

$$\frac{\partial j^\mu}{\partial x^\mu} = 0. \tag{2.17}$$

To construct the quantum theory it is important to consider the Hamiltonian formulation. The prescription

$$H = \vec{p}\cdot\vec{v} - L \tag{2.18}$$

together with our formula for $\vec{p}$ and L leads to the *numerical* expression

$$H = \frac{m}{\sqrt{1-v^2}} + qA_0 \tag{2.19}$$

but to consummate the formalism we must eliminate v in favor of p. Fortunately, by grouping the momentum with potential and squaring we find

$$\frac{1}{1-v^2} - 1 = \frac{(\vec{p} - q\vec{A})^2}{m^2} \tag{2.20}$$

and after some simple algebra we derive

$$H = \sqrt{m^2 + (\vec{p} - q\vec{A})^2} + qA_0. \tag{2.21}$$

The appearance of square root here leads to difficulties in quantization. In order to implement the commutation relations, or (more heuristically) wave-particle duality, one would like to make the substitution $\vec{p} \to -i\vec{\partial}$ and implement the Schrödinger wave equation $H\psi = i\frac{\partial\psi}{\partial t}$. But that substitution renders the Hamiltonian, with its square root, into a nasty, nonlocal operator. One could try to work with the square of the Hamiltonian, but then there would be negative as well as positive energy solution. It was considerations like these that lead Dirac to Dirac equation.

Since an adequate treatment of relativistic quantum theory in any case seems to require that we pass from a point particle to a quantized field description of the charge matter, for present purposes it is preferable to retreat to the non-relativistic limit of H,

$$H = m + \frac{(\vec{p} - q\vec{A})^2}{2m} + qA_0. \tag{2.22}$$

Now there is no difficulty in making the above-mentioned substitution and formulating the Schrödinger wave equation, which becomes

$$i\frac{\partial\psi}{\partial t} = (m + \frac{(-i\vec{\partial} - q\vec{A})^2}{2m} + qA_0)\psi \tag{2.23}$$

(Of course, the constant term m on the right hand side can be eliminated by absorbing a factor e^{-imt} into ψ.)

For the gauge transformation $A'_\mu = A_\mu + \partial_\mu \chi$ on the potentials to leave the Schrödinger equation invariant, it must be accomplished by the transformation

$$\psi' = e^{-iq\chi}\psi \tag{2.24}$$

on the charged fields. Note that the value of charge q appears explicitly in this transformation law. Note also that the *complex* nature of the field is essential, since the transformation involves multiplication by a phase factor.

2.1 Gauge Symmetry for Quantum Fields

Since quantum fields associated with charged particles can create one-particle states whose wave-functions obey the single-particle Schrödinger equation, we should expect that under a gauge transformation such fields transform as in Equation 2.15.

It is possible and proves fruitful to consider reversing the logic, by taking the symmetry law as primary. Indeed, suppose that we have several quantum fields $\psi_1, \psi_2, \cdots$ with charges $q_1, q_2, \cdots$. Then the condition that an interaction term of the general form

$$\Delta L = \psi_1^{p_1} \psi_2^{p_2} \cdots \tag{2.25}$$

which may also contain derivations, should conserve charge is that

$$p_1 q_1 + p_2 q_2 + \cdots = 0 \tag{2.26}$$

for the term destroys p_1 particles charge q_1, p_2 particles charge q_2, and so forth (or creates an equivalent number of anti-particles). Putting it differently, the necessary and sufficient condition for charge conservation is invariance under the symmetry

$$\psi_n' = e^{-iq_n\chi} \tag{2.27}$$

where χ is a real number variable. (Complex numbers are not allowed. They would resale the magnitude of the fields. While they might leave the classical action invariant, they generally spoil the commutation relations.) If there are derivative terms, we cannot allow χ to be a field depending on space and time, because the derivatives $\partial_\mu \psi_n$ transform as

$$\partial_\mu \psi_n' = e^{-iq_n\chi}(\partial_\mu \chi_n - iq_n \partial_\mu \chi \psi_n). \tag{2.28}$$

The complicated inhomogeneous form of this transformation law makes it difficult to form invariant interaction terms containing derivatives. But in order to obtain sensible equations of motion it is necessary to have derivative terms in the action. So if we want to promote charge conservation to]it local gauge symmetry, permitting transformations in which χ depends on space and time, we must modify the derivatives.

A suitable modification is suggested by our earlier discussion of the point particle Hamiltonian. We define the covariant derivative operate D_μ to act on a field ψ_n of charge q_n according to

$$D_\mu \psi_n = (\partial_\mu + iq_n A_\mu)\psi_n. \tag{2.29}$$

Then if A_μ transforms as in Equation 2.15, we have

$$\begin{aligned} D'_\mu \psi'_n &= (\partial_\mu + iq_n A'_\mu) e^{-iq_n \chi} \psi_n \\ &= e^{-iq_n \chi} (\partial_\mu - iq_n \partial_\mu \chi + iq_n A_\mu + iq_n \partial_\mu \chi) \psi_n \\ &= e^{-iq_n \chi} D_\mu \psi_n. \end{aligned} \tag{2.30}$$

Thus the covariant derivative of ψ_n transforms in the same way as ψ_n itself, and there is no difficulty in using it to form invariant interaction terms.

Chapter 3

Grand Unified Theory

3.1 SU(5) Unification

Gauge bosons:

$$\left(\begin{array}{c|c} SU(3) & \\ \hline & SU(2) \end{array}\right) \tag{3.1}$$

$U(1)$: (commuting with $SU(3) \times SU(2)$)

$$\begin{pmatrix} e^{2i\lambda} & & & & & \\ & e^{2i\lambda} & & & & \\ & & e^{2i\lambda} & & & \\ & & & e^{-3i\lambda} & & \\ & & & & e^{-3i\lambda} & \end{pmatrix} \tag{3.2}$$

Or Lie algebra:

$$\begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & -3 & & \\ & & & & -3 & \end{pmatrix} \tag{3.3}$$

One gets breaking $SU(5) \to SU(3) \times SU(2) \times U(1)$ in this pattern with an adjoint "Higgs" field obeying

$$< \Phi > \propto \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{3.4}$$

What do the functions think of this?

$$\begin{pmatrix} u \\ d \end{pmatrix}_{L \;\; \frac{1}{6}}, \begin{pmatrix} v \\ e \end{pmatrix}_{L \;\; -\frac{1}{2}} \tag{3.5}$$

$$\begin{array}{cc} u_{R \;\; \frac{2}{3}} & (u_R^C)_{- \;\; \frac{2}{3}} \\ d_{R \;\; -\frac{1}{3}} \overset{change \; conjugation}{\longrightarrow} & (d_R^C)_{\frac{1}{3}} \\ e_{R \;\; -1} & (e_R^C)_1 \end{array} \tag{3.6}$$

all L – handed .
Multiplets? Clues:

$$15 = \underbrace{10}_{antisymmetric \; tensor} + \underbrace{5}_{vector} \tag{3.7}$$

$$\sum Y = 0 \tag{3.8}$$

Altogether?

$$6 \times \frac{1}{6} + 2 \times -\frac{1}{2} + 3 \times -\frac{2}{3} + 3 \times \frac{1}{3} + 1 \quad = \quad 0 \tag{3.9}$$

$$(d_R^C)_{\frac{1}{3}} + \begin{pmatrix} v \\ e \end{pmatrix}_{L \;\; -\frac{1}{2}} \quad make \;\; 5 \tag{3.10}$$

Actually (using $\sum_{\alpha\beta}$)

$$\begin{pmatrix} d_R^C \\ -e \\ v \end{pmatrix} = \bar{5} \tag{3.11}$$

Note:

$$\tilde{g} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} = g' Y \tag{3.12}$$

with

$$\tilde{g} = \frac{-g'}{6} \tag{3.13}$$

Can the residue be identified with 10? $\Psi^{\alpha i}, \alpha = 1,2,3, i = 4,5$, color: 3, $SU(2) : 2$.

$$Y = -\frac{1}{6}(2-3) = \frac{1}{6} \Rightarrow \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{3.14}$$

$\Psi^{\alpha\beta}$, color: $\bar{3}$, $SU(2)$: singlet.

$$Y = -\frac{1}{6}(2+2) = -\frac{2}{3} \Rightarrow u_R^C \tag{3.15}$$

Ψ^{ij}, color: singlet, $SU(2) : 2$

$$Y = -\frac{1}{6}(-3-3) = 1 \Rightarrow e_R^C \tag{3.16}$$

It clicks.
Normalizing $\tilde{g}$:

$$SU(3)\ generators \quad : \quad g_{un.} \begin{pmatrix} -\frac{1}{2} & & & & & \\ & -\frac{1}{2} & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \end{pmatrix} etc. \tag{3.17}$$

$$SU(2)\ generators \quad : \quad g_{un.} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & \frac{1}{2} & & \\ & & & & -\frac{1}{2} & \end{pmatrix} etc. \tag{3.18}$$

$$tr\Gamma_a\Gamma_b \quad = \quad \frac{1}{2} f_{ab} \tag{3.19}$$

$$U(1)\ generators \quad : \quad \tilde{g} \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$$

$$= g' \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix} \tag{3.20}$$

$$g'^2(3 \cdot (\frac{1}{3})^2 + 2 \cdot (\frac{1}{2})^2) = \frac{1}{2} g^2_{un.} \tag{3.21}$$

$$g'^2(\frac{5}{6}) = \frac{1}{2} g^2_{un.} \tag{3.22}$$

$$g'^2 = \frac{3}{5} g^2_{un.} \tag{3.23}$$

So "naive" prediction:

$$g_S^2 = g_\omega^2 = \frac{5}{3} g^2_{un.} \tag{3.24}$$

$$\sin^2 \theta_\omega = \frac{g'^2}{g'^2 + g_\omega^2} = \frac{\frac{3}{5}}{1 + \frac{3}{5}} = \frac{3}{8} \tag{3.25}$$

Expt. ≈ 0.22.

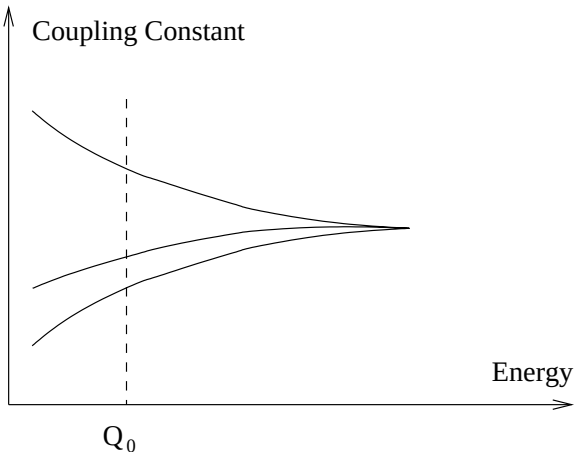

Figure 3.1: Coupling Constant

Need substantial remaining. Q_0 = observation point (e.g. M_2).
Constraints: 3 observable $\rightarrow$ 2 quantities.

$$\frac{dg_i}{dt} = \beta_i(g_i) \tag{3.26}$$

$$\frac{d\frac{1}{g_i^2}}{dt} = -2\beta_i^\circ \tag{3.27}$$

$$\frac{1}{g_3^2(Q_0)^2} - \frac{1}{g_3^2(M_{un.})} = 2\beta_3^\circ \ln\frac{M_U}{Q_0} \tag{3.28}$$

$$\frac{1}{g_i(Q_0)^2} - 2\beta_i^\circ \ln\frac{M_U}{Q_0} \quad : \quad independent\ of\ i \tag{3.29}$$

$$\frac{\frac{1}{g_i(Q_0)^2} - \frac{1}{g_j(Q_0)^2}}{\beta_i^\circ - \beta_j^\circ} \quad : \quad independent\ of\ i,j \tag{3.30}$$

N.B.: of course

$$g_1^2 = \frac{5}{3}g'^2 \tag{3.31}$$

Master formula:

$$\beta_0 = -\frac{11}{3}e_2 + \frac{4}{3}T_{\frac{1}{2}} + \frac{1}{3}T_0 \ \begin{cases} real \times \frac{1}{2} \\ weyl \times \frac{1}{2} \end{cases} \tag{3.32}$$

Minimal Standard Model (MSM):

$$\beta^{(3)} = -11 + \frac{4}{3}\times 6 \times \frac{1}{2} = -7 \tag{3.33}$$

$$\beta^{(2)} = -\frac{11}{3}\times 2 + \frac{4}{3}\times 12 \times \underbrace{\frac{1}{2}}_{weyl} \times\frac{1}{2} + \frac{1}{3}\times\frac{1}{2} = -\frac{19}{6} \tag{3.34}$$

$$\begin{aligned}\beta^{(1)} = {} & \frac{3}{5}\{\frac{4}{3}\times \underbrace{3}_{families} \times \underbrace{\frac{1}{2}}_{weyl} [6\times(\frac{1}{6})^2 + 3\times(\frac{2}{3})^2 + 3(\frac{1}{3})^2 + 2\times(\frac{1}{2})^2 + 1^2] \\ & + \frac{1}{3}\times 2\cdot(\frac{1}{2})^2\} = \frac{41}{10}\end{aligned} \tag{3.35}$$

MSSM:

$$\Delta\beta^{(3)} = \frac{4}{3}\times 3 \times \underbrace{\frac{1}{2}}_{real} + \frac{1}{3}\times 12\times\frac{1}{2} = 4 \tag{3.36}$$

2 Higgs + Higgsions:

$$\Delta\beta^{(2)} = \begin{array}{ll} \frac{4}{3}\times 2\times\frac{1}{2} & Gaugions \\ +\frac{4}{3}\times 2\times\frac{1}{2}\times\frac{1}{2} & Higgsions \\ +\frac{1}{3}\times 12\times\frac{1}{2} & Sferminos \\ +\frac{1}{3}\times\frac{1}{2} & Extra\ Higgs \end{array}$$

$$= \frac{25}{6} \tag{3.37}$$

Higgsions:

$$\begin{aligned}\Delta\beta^{(1)} &= \frac{3}{5}\{\frac{4}{3}\times 2\times(2\times(\frac{1}{2})^2)\times \underbrace{\frac{1}{2}}_{weyl} \\ &\quad +\frac{1}{3}\times 3[6\times(\frac{1}{6})^2+3\times(\frac{2}{3})^2+3\times(\frac{1}{3})^2+2\times(\frac{1}{2})^2+1] \\ &\quad +\frac{1}{3}\times(\frac{1}{2})^2\} \\ &= \frac{5}{2}\end{aligned} \tag{3.38}$$

Leaving out Higgs:
MSM:

$$\beta^{(3)} = -7 \tag{3.39}$$

$$\beta^{(2)} = -\frac{10}{3} \tag{3.40}$$

$$\beta^{(1)} = 4 \tag{3.41}$$

MSSM:

$$\Delta\beta^{(3)} = 4 \tag{3.42}$$

$$\Delta\beta^{(2)} = \frac{10}{3} \tag{3.43}$$

$$\Delta\beta^{(1)} = 2 \tag{3.44}$$

$$\frac{\Delta\beta^{(3)}-\Delta\beta^{(2)}}{\beta^{(3)}-\beta^{(2)}} = -\frac{2}{11} \tag{3.45}$$

$$\frac{\Delta\beta^{(3)}-\Delta\beta^{(1)}}{\beta^{(3)}-\beta^{(1)}} = -\frac{2}{11} \tag{3.46}$$

$$\frac{\Delta\beta^{(2)}-\Delta\beta^{(1)}}{\beta^{(2)}-\beta^{(1)}} = -\frac{2}{11} \tag{3.47}$$

Therefore, all correlations to unification come from one doublet $\Rightarrow$ 6 effective doublets.

Also change in M_U:

$$\ln \frac{M_U}{Q} \propto \frac{1}{\beta_i - \beta_j} \tag{3.48}$$

$$(\frac{1}{\beta_i - \beta_j})_{SUSY} \simeq (\frac{1}{\beta_i - \beta_j})_{MSM}(1 - \frac{2}{11}) \simeq \frac{11}{9}(\frac{1}{\beta_i - \beta_j})_{MSM} \tag{3.49}$$

$$\frac{10^{17}}{10^2} \rightarrow 10^{12 \cdot \frac{11}{9} + 2} \simeq 10^{16.5} \tag{3.50}$$

3.2 $SU(5)$

- Fermion Multiplets

$$SU(3) \times SU(2) \times U(1) \subset SU(5) \tag{3.51}$$

Needs $U(1)$ traceless.

LH fields:

	3	2	1	d	$\sum Y$
Q_L	3	2	$\frac{1}{6}$	6	1
CU_R	$\bar{3}$	1	$-\frac{2}{3}$	3	-2
CD_R	$\bar{3}$	1	$\frac{1}{3}$	3	1
L_L	1	2	$-\frac{1}{2}$	2	-1
CE_R	1	1	1	1	1

(3.52)

Look for $5, 10$ with $\sum Y = 0$.

$$F_\mu : \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} & -\frac{1}{2} \\ & C d_R & & e & \nu \end{pmatrix} \tag{3.53}$$

$T^{\mu\nu}$: antisymmetric

$$\begin{array}{lcccc} T^{\alpha\beta} & : & \bar{3}, & 1, & -\frac{2}{3} \\ T^{\alpha i} & : & 3, & 2, & \frac{1}{6} \\ T^{ij} & : & 1, & 1, & 1 \end{array} \tag{3.54}$$

- Symmetry Breaking

Adjoint (traceless)

$$\begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} M \tag{3.55}$$

$$SU(5) \to SU(3) \times SU(2) \times U(1) \tag{3.56}$$

$$\left(\begin{array}{c|c} SU(3) & \\ \hline & SU(2) \end{array}\right) \tag{3.57}$$

Still need $SU(2) \times U(1) \Rightarrow U(1)$.

Minimal:

$$5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{3.58}$$

There could be others.

Mass terms:

$$\varphi^{\mu} T^{\nu\rho} T^{\sigma\tau} \epsilon_{\mu\nu\rho\sigma\tau} \tag{3.59}$$

Note on "Majorana" mass terms:

$$\begin{aligned}
\Psi'\Psi &= (e\bar{\Psi}')\Psi && (3.60)\\
\Psi_L &= (\frac{1-\gamma_5}{2})\Psi && (3.61)\\
(\frac{1-\gamma_5}{2})\Psi_L &= 0 && (3.62)\\
\gamma_5 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} && (3.63)\\
\Psi_L &= \begin{pmatrix} \eta \\ -\eta \end{pmatrix} && (3.64)\\
C\Psi' &= \gamma_2 \Psi'^{*} && (3.65)\\
C\Psi'_2 &= \begin{pmatrix} i\sigma_2\eta^* \\ i\sigma_2\eta^* \end{pmatrix} && (3.66)\\
(e\bar{\Psi}') &= \begin{pmatrix} i\sigma_2\eta' & i\sigma_2\eta' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \eta \\ -\eta \end{pmatrix} && (3.67)\\
&\propto \eta'_a \epsilon^{ab} \eta_b && (3.68)
\end{aligned}$$

where, a and b are Dirac indices.

Therefore,

$$\eta_a^{(i)'} \epsilon^{ab} \eta_b^{(j)} \tag{3.69}$$

Note symmetric in $i \Leftrightarrow j$, due to Fermi statistics.

Link between texture and representation at unification: (g_{ab} symmetric)

$$\varphi^{\mu} T^{(a)\nu p} T^{(b)\sigma\tau} \tag{3.70}$$

allows self-mass for 2 component neutral fields.

$$\frac{\upsilon}{\sqrt{2}} \epsilon_{5\alpha\beta\gamma ji} \underbrace{T^{\alpha\beta} T^{\gamma i}}_{CU_R U_L} \tag{3.71}$$

$\varphi^*_{\mu} T^{\mu\nu} F_{\nu}$:

$$\frac{\upsilon}{\sqrt{2}} T^{5\alpha} F_{\alpha} \quad : \quad D_L \; C D_R \tag{3.72}$$

$$\frac{\upsilon}{\sqrt{2}} T^{54} F_4 \quad : \quad C E_R \; E_L \tag{3.73}$$

- B Validation

  Vector bosons:

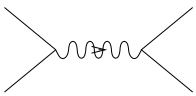

Figure 3.2: Vectro Bosons

$$T^*_{12} T^{14} X^2_4 X^4_2 F^{*2} F_4 \tag{3.74}$$

$$u^3_R u'_L \quad d^2_R e_L \tag{3.75}$$

Triplet Higgs:

$$(\epsilon_{12345} T^{12} T^{45} \varphi^3)(\varphi^*_3 T^{31} F_1) \tag{3.76}$$

$$\bar{u_{R_3}} \bar{e_R} \bar{u_{R_2}} \bar{d_{R_1}} \tag{3.77}$$

Single appearance of $\epsilon_{\alpha\beta\gamma}$.

Phenomenology $\Rightarrow M$ large.

- Implementing SB; Hierarchy problem $\varphi^+\varphi$, $\varphi^+ A\varphi$, $\varphi^+ A^2\varphi$, $trA^2\varphi^+\varphi$, trA^2, trA^3, trA^4, $tr(A^2)^2$. Need big vev for A, small for φ. Heavy φ^{α} not by decoupling, but by conspiracy. Not inconsistent, but ugly.

- Normalization

$$g \begin{pmatrix} \cdots & & \\ & \frac{1}{2} & \\ & & \frac{1}{2} \end{pmatrix} \tag{3.78}$$

$$g' \begin{pmatrix} \frac{1}{3} & & & & & \\ & \frac{1}{3} & & & & \\ & & \frac{1}{3} & & & \\ & & & -\frac{1}{2} & & \\ & & & & -\frac{1}{2} & \end{pmatrix} = g()_{\sum d_i^2 = \frac{1}{2}} \tag{3.79}$$

$$g'^2(3 \cdot (\frac{1}{3})^2 + 2 \cdot (\frac{1}{2})^2) = g^2 \frac{1}{2} \tag{3.80}$$

$$g'^2(\frac{5}{6}) = g^2 \frac{1}{2} \tag{3.81}$$

$$g'^2 = \frac{3}{5} g^2 \tag{3.82}$$

$$\sin^2 \theta_w = \frac{g'^2}{g^2 - g'^2} = \frac{\frac{3}{5} g^2}{(1 + \frac{3}{5}) g^2} = \frac{3}{8} \tag{3.83}$$

Expct. ≈ 0.22.

Also, of course,

$$\frac{g_{SU(2)}}{g_{SU(3)}} = 1 \tag{3.84}$$

3.3 $SO(10)$ Unification

$SU(6)$?

$$\frac{6\times 5}{2}, F^{ab}, T^{\mu\nu} \tag{3.85}$$

$SU(5)$ in $SO(10)$: 5 complex components

$$\begin{aligned} Z_j &= X_j + iY_j & (3.86)\\ < Z'|Z > &= \underbrace{\sum X'_j X_j + Y'_j Y_j}_{SO(10)\ leaves\ this\ part\ invariant} + i \underbrace{\sum X'_j Y_j - Y'_j X_j}_{SP(10)\ leaves\ this\ part\ invariant} & (3.87)\end{aligned}$$

$SO(10)$ commutators (structure contents): rotation in kl plane

$$\begin{aligned} \delta^{kl}_{\epsilon} X_j &= \epsilon(\delta_{jk}X_l - \delta_{jl}X_k) & (3.88)\\ (\delta^{kl}_{\epsilon}\delta^{mn}_{\eta} - \delta^{mn}_{\eta}\delta^{kl}_{\epsilon})X_j &= \eta\delta^{kl}_{\epsilon}(\delta_{jm}X_n \\ &\quad -\delta_{jn}X_m) - \epsilon\delta^{mn}_{\eta}(\delta_{jk}X_l - \delta_{jl}X_k) & (3.89)\\ &= \epsilon\eta\{\delta_{jm}\delta_{nk}X_l - \delta_{jm}\delta_{nl}X_k \\ &\quad -\delta_{jn}\delta_{mk}X_l + \delta_{jn}\delta_{ml}X_k \\ &\quad -\delta_{jk}\delta_{lm}X_n - \delta_{jk}\delta_{ln}X_m \\ &\quad +\delta_{jl}\delta_{km}X_n - \delta_{jl}\delta_{kn}X_m\} & (3.90)\\ &= \delta_{nk}T^{lm} - \delta_{nl}T^{km} - \delta_{mk}T^{ln} + \delta_{ml}T^{kn} & (3.91)\end{aligned}$$

Γ matrices "=" $\sqrt{rotation}$ (spinor rep.)

$$\{\Gamma_k, \Gamma_l\} = 2\delta_{kl} \tag{3.92}$$

Claim: $-\frac{1}{4}[\Gamma_k, \Gamma_l]$ satisfy the $SO(10)$ commutators.

$$[\Gamma_k, \Gamma_l] = 2(\Gamma_k\Gamma_l - \delta_{kl}) \tag{3.93}$$

So

$$\begin{aligned} \frac{1}{16}[[\Gamma_k, \Gamma_l], [\Gamma_m, \Gamma_n]] &= \frac{1}{4}[\Gamma_k\Gamma_l, \Gamma_m\Gamma_n] & (3.94)\\ &= \frac{1}{4}(\Gamma_k\Gamma_l\Gamma_m\Gamma_n - \Gamma_m\Gamma_n\Gamma_k\Gamma_l) & (3.95)\end{aligned}$$

Now use

$$\Gamma_a\Gamma_b = -\Gamma_b\Gamma_a + 2\delta_{ab} \tag{3.96}$$

to pull through T_k and T_l in turn. End terms cancel.
Pick up terms:

$$\frac{1}{4}(\Gamma_k\Gamma_l\Gamma_m\Gamma_n - \Gamma_m\Gamma_n\Gamma_k\Gamma_l) = \frac{1}{2}(-\delta_{nk}\Gamma_m\Gamma_l + \delta_{mk}\Gamma_n\Gamma_l - \delta_{nl}\Gamma_k\Gamma_m + \delta_{ml}\Gamma_k\Gamma_n) \tag{3.97}$$

Now use

$$\Gamma_m\Gamma_l = \frac{1}{2}([\Gamma_m, \Gamma_l] + 2\delta_{ml}) \quad etc. \tag{3.98}$$

$\delta\delta$ terms all cancel:

$$\begin{gathered}\frac{1}{2}(-\delta_{nk}\Gamma_m\Gamma_l + \delta_{mk}\Gamma_n\Gamma_l - \delta_{nl}\Gamma_k T_m + \delta_{ml}\Gamma_k\Gamma_n) = \\ \frac{1}{4}(-\delta_{nk}[\Gamma_m, \Gamma_l] + \delta_{mk}[\Gamma_n, \Gamma_l] - \delta_{nl}[\Gamma_k, \Gamma_m] + \delta_{ml}[\Gamma_k, \Gamma_n])\end{gathered} \tag{3.99}$$

Compare to

$$\delta_{nk}\Gamma^{lm} - \delta_{nl}\Gamma^{km} - \delta_{mk}\Gamma^{ln} + \delta_{ml}\Gamma^{kn} \tag{3.100}$$

QED.

$$U^{-1}(R)T^\mu U(R) = R^\mu_\nu T^\nu \tag{3.101}$$

Construction of Γ matrices:

$$\begin{aligned}
\Gamma_1 &= \sigma_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 && (3.102)\\
\Gamma_2 &= \sigma_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1 && (3.103)\\
\Gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1 \otimes 1 && (3.104)\\
\Gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes 1 \otimes 1 \otimes 1 && (3.105)\\
\Gamma_5 &= \sigma_3 \otimes \sigma_2 \otimes \sigma_1 \otimes 1 \otimes 1 && (3.106)\\
&\vdots
\end{aligned}$$

with Pauli σ – matrices does the job.
Note:

$$R_{12} = \frac{i}{2}\sigma_3 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \tag{3.107}$$

$$R_{34} = \frac{i}{2}1 \otimes \sigma_3 \otimes 1 \otimes 1 \otimes 1 \tag{3.108}$$

$$\vdots$$

So we diagonalize:

$$SO(2) \otimes SO(2) \otimes SO(2) \otimes SO(2) \otimes SO(2) \subset SO(10) \tag{3.109}$$

This gives us a $2^5 = 32$ – dimensional representation of $SO(10)$ by

$$R(e^{i\theta_{ab}T_{ab}}) = e^{i\theta_{ab}(-\frac{1}{4}[\Gamma_a,\Gamma_b])} \tag{3.110}$$

It is not quite irreducible.
Note

$$K = -i\Gamma_1\Gamma_2\cdots\Gamma_{10} \tag{3.111}$$

anticommutes with all the Γ_i.
Also $K^* = K$ and K is Hermitean (exercise).

$$K^2 = -i\Gamma_i\cdots\Gamma_{10}\Gamma_i\cdots\Gamma_{10} = \underbrace{(-)^{1+\frac{10\times 9}{2}}}_{pulling\ through} = 1 \tag{3.112}$$

Therefore, the R_{kl} commute with K and we can project spinos.

$$S \to \frac{1+K}{2}S \tag{3.113}$$

These make the $16 + 16'$ representations, which are irreducible.
Shift-register rotation: The components of S can be labeled by their $SO(2)^5$ eigenvalues $\pm\frac{1}{2}$ (supplying a $-i$).

$$R_{before} = iR_{standard} \tag{3.114}$$

Since

$$k = +\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \tag{3.115}$$

the 16 has even number of signs, $\overline{(16)}$ has odd number of signs.
Back to $SU(5)$, $v'Jv$ invariant, with

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} = T_{12} + \cdots + T_{910} \tag{3.116}$$

$$\upsilon \rightarrow \upsilon + \epsilon G\upsilon \tag{3.117}$$

$$\Delta\upsilon' J\upsilon = \epsilon\upsilon'(G^T J + JG)\upsilon \tag{3.118}$$

$$= \epsilon\upsilon'(-GJ + JG)\upsilon \tag{3.119}$$

$$\Rightarrow_{spinor} \sum \sigma_2 \; invariant \tag{3.120}$$

Similarly we identify

$$SU(3) \subset SO(6) \tag{3.121}$$

$$SU(2) \subset SO(4) \tag{3.122}$$

and an extra $U(1)$ for hyperchrnge. The diagonal elements (maximal torus) are simply represented as the $R_{2i-1,2i}(-trace\ part)$. Thus,

$$R_{12} - \frac{1}{3}(R_{12} + R_{34} + R_{56})\ etc. \in SU(3) \tag{3.123}$$

$$R_{78} - R_{910} \in SU(2) \tag{3.124}$$

With

$$\frac{1}{6}(R_{12} + R_{34} + R_{56}) - \frac{1}{4}R_{78} - R_{910} \propto U(1)_Y \tag{3.125}$$

Analysis of 16 standard model $Q - \#s$:
$5 + signs$

$$1\ state$$
$$|+++++>$$
$$SU(3) \times SU(2) \times U(1)\ singlet \tag{3.126}$$

$1 + signs$

$$5\ state, 2\ types$$
$$|--+--> $$

$$
\begin{array}{r}
|-+--->\\
\underbrace{|+----->}\\
SU(3)\ \overline{(3)}\\
SU(2)\ singlet
\end{array}
$$

$$\tilde{Y} = \frac{1}{6}(-1) - \frac{1}{4}(-2) = \frac{1}{3} \tag{3.127}$$

$$
\begin{array}{r}
|---+->\\
\underbrace{|----+>}\\
SU(3)\ singlet\\
SU(2)\ doublet
\end{array}
$$

$$\tilde{Y} = \frac{1}{6}(-3) = -\frac{1}{2} \tag{3.128}$$

With $Y = \tilde{Y}$, these can be identified as

$$d_R^C, \begin{pmatrix} -e \\ \nu \end{pmatrix}_L \tag{3.129}$$

They are our $\bar{5}$ of $SU(5)$.
$3+$ *signs*

$$
\begin{array}{r}
10\ state, 3\ types\\
|+++-->\\
SU(3)\ singlet\\
SU(2)\ singlet
\end{array}
$$

$$Y = \frac{1}{6}(+3) - \frac{1}{4}(-2) = 1 \equiv e_R^C \tag{3.130}$$

$$
\begin{array}{r}
|+--++>\\
|-+-++>\\
\underbrace{|+--++>}\\
SU(3)\ \bar{3}\\
SU(2) singlet
\end{array}
$$

$$Y = \frac{1}{6}(-1) - \frac{1}{4}(2) = -\frac{2}{3} \equiv u_R^C \tag{3.131}$$

$$
\begin{array}{r}
|++-+->\\
|+-++->
\end{array}
$$

$$
\begin{aligned}
&|-+++-> \\
&|++--+> \\
&|+-+-+> \\
&|-++-+> \\
&\qquad SU(3)\ 3 \\
&\qquad SU(2)\ 2
\end{aligned}
$$

$$Y = \frac{1}{6}(1) = \frac{1}{6} \equiv \begin{pmatrix} u \\ d \end{pmatrix}_L \tag{3.132}$$

Comments:

1. The construction of Γ –matrices – anticommuting objects – used here contains the existance of bosonization in $1+1d$ field theories. Also, the fermionization of spin chains (from $\sigma \otimes \sigma \otimes \cdots \rightarrow$ anticommuting quantities), known as Jordan-Wigner trick.

2. $$|+++++> \equiv N_R^C \tag{3.133}$$

   plays an important role in current thinking about symmetric masses. It can get a mass (Majorana Mass) of the type

   $$M_{ij}\bar{N}_R^i N_R^{Cj} \tag{3.134}$$

   This involves breaking $SO(10)$, but not $SU(3) \times SU(2) \times U(1)$, so M can be large. Also, N_S can connect to ordinary left-handed neutrinos through the ordinary Higgs doublet, in the form

   $$\mu_{ij}\bar{N}_R^i L^{\mu j}\varphi_\mu^* \tag{3.135}$$

   where, i, j are formly indices and $\mu = SU(2)$ index.

   By 2^{nd} order perturbation theory we induce Majorana Masses for the ν_L.

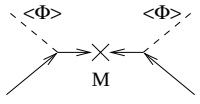

Figure 3.3: Majorana Masses

$$m \sim \frac{\mu^2}{M} \tag{3.136}$$

Breaking Scheme: Higgs φ in 16

$$\begin{aligned} <\varphi_N> &\neq 0 && (3.137)\\ SO(10) &\to SU(5) && (3.138)\end{aligned}$$

with "right" hyperchrnge.
Extra $U(1)$:

$$\begin{array}{lcc} N_R^C: & 5 & 5 \\ d_R^C, L_L: & -3 & -15 \\ u_R^C, Q_L, e_R^C: & 1 & 10 \end{array} \qquad (3.139)$$

$$\alpha B + \beta L + \gamma Y \qquad (3.140)$$

N_R^C:

$$\begin{aligned} -\beta &= 5 \\ \beta &= -5 && (3.141)\end{aligned}$$

d_R^C:

$$\begin{aligned} -\frac{\alpha}{3} + \frac{\gamma}{3} &= -3 \\ \alpha &= 5 && (3.142)\end{aligned}$$

L_L:

$$\begin{aligned} \beta - \frac{\gamma}{2} &= -3 \\ \gamma &= -4 && (3.143)\end{aligned}$$

u_R^C:

$$-\frac{5}{3} - \frac{2\gamma}{3} \stackrel{?}{=} 1 \qquad (3.144)$$

$$-\frac{5}{3} + \frac{\gamma}{3} = 1 \qquad (3.145)$$

Q_L:

$$\begin{aligned} \frac{\alpha}{3}+\frac{\gamma}{6} &= \frac{5}{3}+\frac{2}{6} \\ &= 1 \end{aligned} \tag{3.146}$$

e_R^C:

$$-\beta+\gamma=1 \tag{3.147}$$

$$\alpha B+\beta L+\gamma Y = 5B-5L-4Y \tag{3.148}$$

There are $45_{adjoint}$ and 10_{vectro} as before.
Fermion masses: bilinears in φ

$$\Gamma_1 = \sigma_1\otimes 1\otimes\cdots\otimes 1 \tag{3.149}$$
$$\Gamma_2 = \sigma_2\otimes 1\otimes\cdots\otimes 1 \tag{3.150}$$
$$\vdots$$

φ^* transforms as $e^{[\Gamma_\mu,\Gamma_\nu]^*}$

$$\begin{aligned} (C\varphi^*)' &= Ce^{[\Gamma_\mu,\Gamma_\nu]^*}\varphi^* \\ &= e^{[\Gamma_\mu,\Gamma_\nu]^*}C\varphi^* \end{aligned} \tag{3.151}$$
$$C\Gamma_\mu^* = \pm\Gamma_\mu C \tag{3.152}$$
$$C = \Gamma_1\Gamma_3\Gamma_5\Gamma_7\Gamma_9 \tag{3.153}$$

With + sign:

$$C=\sigma_1\otimes i\sigma_2\otimes\sigma_1\otimes i\sigma_2\otimes\sigma_1 \tag{3.154}$$

symmetric and real.
Anticommute with K: Construct Majorana bilinears

$$\varphi\varphi\sim(\varphi^*C)^+\ tensor\ \overrightarrow{made}\ from\ \Gamma\ \varphi \tag{3.155}$$

to be consistent with projection add number of indices. Irreducible $\Rightarrow$ Totally antisymmetric.

$$16\times 16 = \underbrace{10_{vector}}_{S}+\underbrace{120_{3-tensor}}_{A}+\underbrace{126_{5-tensor:self-dual}}_{S} \tag{3.156}$$
$$16\times\overline{16} = 1_{scale}+45_{2-tensor}+210_{4-tensor} \tag{3.157}$$

Final comments on $SO(10)$ vs. $SU(5)$: 3 RH neutrinos vs. $SU(5)\times U(1)$. Charge quantization.

$$Q \propto \epsilon(B-L) \tag{3.158}$$

$p : 1+\epsilon$, $e : -1-\epsilon$, $n : \epsilon$, $v : -\epsilon$.

N.B.: mechanics of charge quantization. $n \to pe\bar{v}$.

Chapter 4

Renormalization

Cut off dependence in a few basic quantities this into redefinition:

$$
\begin{aligned}
L &= -\frac{1}{4}\bar{t}^{a}_{\mu\nu 0}\bar{t}_{0}^{a\mu\nu} + \bar{\psi}_0(ip_0 m_0)\psi_0 && (4.1)\\
A_{ren.} &= \frac{1}{\sqrt{Z_A}}A_0 && (4.2)\\
\psi_{ren.} &= \frac{1}{\sqrt{Z_\psi}}\psi_0 && (4.3)\\
Z_{T_{\bar{\psi}\psi A}} g_{ren.}\bar{\psi}_{ren.}\gamma^{\mu}\psi_{ren.}A_{\mu ren.} &= g_0\bar{\psi}_0\gamma^{\mu}\psi_0 A_{\mu 0} && (4.4)\\
g_{ren.} &= \frac{g_0 Z_\psi Z_A^{\frac{1}{2}}}{Z_{T_{\bar{\psi}\psi A}}} && (4.5)\\
Z_{\bar{\psi}\psi}\bar{\psi}_{ren.}\psi_{ren.}m_{ren.} &= \bar{\psi}_0\psi_0 m_0 && (4.6)\\
m_{ren.} &= \frac{m_0 Z_\psi}{Z_{\bar{\psi}\psi}} && (4.7)
\end{aligned}
$$

The wave-function renormalizations are absorbed into normalizing the 1–particle states. The renormalized couplings are physical particles matrix of renormalized states. Indeed, these are the parameters define the theory, at least perturbatively. We fix these to experiment. Then, other quantities must be expressed finite terms using them (and we remove the cut off).

Notice that in a gauge theory, we must use a gauge-invariant regulator to get the simplicity. Otherwise, there are A^2 (no derivatives res is like), $A\sigma A, AAA, A^4$, etc. coupling appearing. We may set them to zero and entree conspiracies to record gauge symmetry (throwing their a gauge invariant regulation).

$$
\begin{aligned}
g_{ren.} &= \frac{Z_A^{\frac{3}{2}}}{\Gamma_{A^3}} g_0 \quad vector\ 3-pt. && (4.8)\\
&= \frac{Z_A}{\sqrt{\Gamma_{A^4}}} g_0 \quad vector\ 4-pt. \propto g^2 && (4.9)\\
&= \frac{Z_A^{\frac{1}{2}} Z_\psi}{\Gamma_{\bar{\psi}\psi A}} && (4.10)\\
&\vdots \\
&= Z && (4.11)\\
\frac{Z_a}{Z_{\bar{\nu}mA}} &= universal\ (Ward\ identity) && (4.12)
\end{aligned}
$$

In Abelian theory are equal 1. In N.A. theory, not generally (+ gauge dependent).

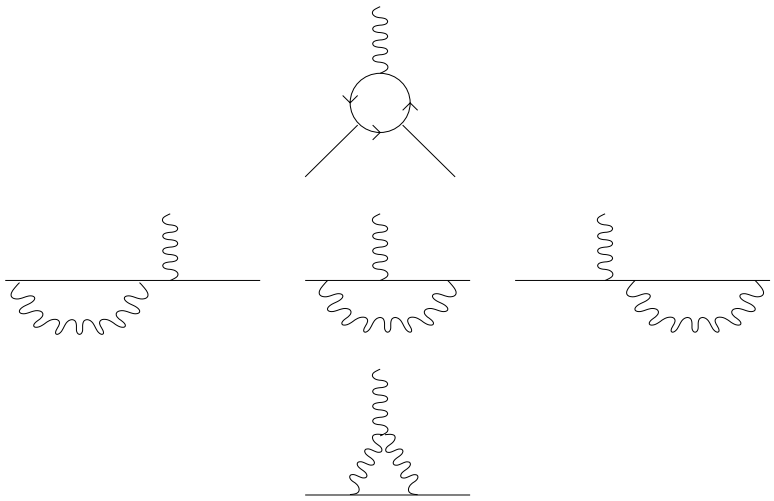

Figure 4.1: Abelian Theory.

To clarify this structure, let's work an example. Correlation function of Bare $\bar{\psi}, \psi$ (inverse propagator).

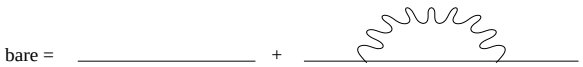

Figure 4.2: Bare.

$$
bare = -i(\not{p} - m) + \int^{\Lambda} \frac{d^4k}{(2\pi)^4} (ie)^2 \frac{(-i)(i)\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{k^2[(p-k)^2 - m^2]} \qquad (4.13)
$$

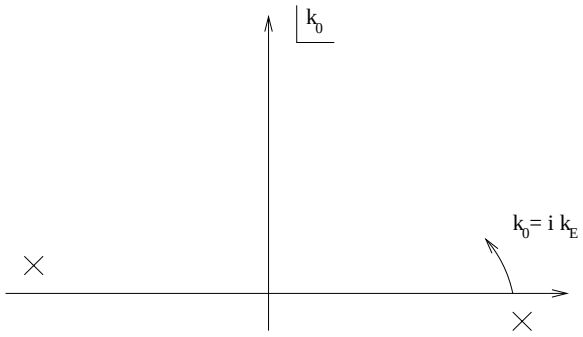

Figure 4.3: Bare Diagram.

Renormalized:

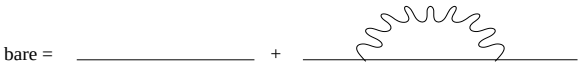

Figure 4.4: Bare.

$$bare = -iZ(\not{p} - m_R) + e_i^2(no\ cut - off) \tag{4.14}$$

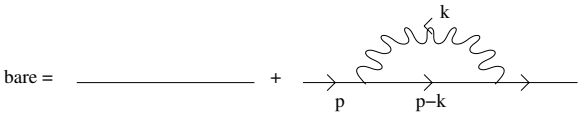

Figure 4.5: Renormalized.

$$\begin{aligned} bare &= -i(\not{p} - m) + ie^2 \int^{\Lambda} \underbrace{\frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(\not{p} - \not{k} + m)\gamma^\mu}{k^2[(p-k)^2 - m^2]}}_{new\ Eulidec} && (4.15) \\ S_3 &= 2\pi^2 && (4.16) \\ bare &= -i(\not{p} - m) + \frac{ie^2}{8\pi^2} \int^{\Lambda} d\Omega dk k^3 \frac{-2(\not{p} - \not{k}) + 4m}{k^2[(p-k)^2 - m^2]} && (4.17) \end{aligned}$$

We are going to keep only the divergent part. We shall subtract by normalizing the log divergence (only) at μ.

$$linear + \log\ divergent \tag{4.18}$$

$$\begin{aligned}
\log \quad : \quad & \int^{\Lambda} d\Omega dk k^3 \frac{-2\not{p} + 4m}{k^2[(p-k)^2 - m^2]} && (4.19)\\
= \quad & \int^{\Lambda} \frac{d\Omega dk k^3}{k^4}[-2\not{p} + 4m] && (4.20)\\
= \quad & \ln\Lambda[-2\not{p} + 4m] && (4.21)\\
linear \quad : \quad & \int^{\Lambda} dr dk k^3 \frac{-2\not{k}}{k^2[(p-k)^2 - m^2]} && (4.22)\\
= \quad & \int^{\Lambda} d\Omega dk k^3 \frac{-2\not{k}}{k^2 - 2pk + p^2 + m^2} && (4.23)\\
= \quad & \int^{\Lambda} d\Omega dk k^3 \frac{-2\not{k}}{k^2(1 - \frac{2pk}{(}k^2) + \frac{p^2}{k^2} + \frac{m^2}{k^2}} && (4.24)\\
= \quad & \int^{\Lambda} d\Omega dk k^3 \frac{-2\not{k}}{k^6} 2pk && (4.25)\\
\int d\Omega \not{k} k_\mu \quad = \quad & \int d\Omega k_\mu k_\nu \gamma_\nu && (4.26)\\
= \quad & \frac{1}{4}\gamma k^2 && (4.27)\\
\int^{\Lambda} d\Omega dk k^3 \frac{-2\not{k}}{k^6} 2pk \quad = \quad & \int^{\Lambda} dk k^3 \frac{\not{p}}{k^4} && (4.28)\\
= \quad & \not{p} \ln\Lambda && (4.29)
\end{aligned}$$

Putting it all together (as discussed):

$$\begin{aligned}
bare \quad = \quad & -i(\not{p} - m_0) - i\frac{e^2}{\gamma\pi^2}\ln\frac{\Lambda}{\mu}[-2\not{p} + 4m_0 + \not{p}] && (4.30)\\
= \quad & iZ_\psi(\not{p} - m_r) && (4.31)\\
Z_\psi \quad = \quad & 1 - \frac{e^2}{8\pi^2}\ln\frac{\Lambda}{\mu} && (4.32)
\end{aligned}$$

Finite matrix $d't'$ at $|p| = \mu$

$$m_R(\mu) = m_0[1 - 3\frac{e^2}{8\pi^2}\ln\frac{\Lambda}{\mu}] \qquad (4.33)$$

Check Ward identify:

$$\begin{aligned}
bare \quad = \quad & -ie\gamma_\mu + \int^{\Lambda} \frac{(-ie)^3\gamma_\nu(\not{p} - \not{k})\gamma_\mu(\not{p} - \not{k})\gamma_\nu(i^2)(-i)}{k^2((k-p)^2)^2} && (4.34)\\
= \quad & -ie\gamma_\mu - \frac{ie^3}{8\pi^2}\int^{\Lambda} d\Omega dk k^3 \frac{-2\not{k}\gamma_\mu\not{k}}{k^6} && (4.35)
\end{aligned}$$

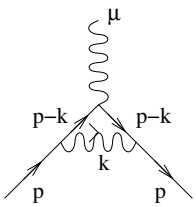

Figure 4.6: Ward Identity.

$$= -ie\gamma_\mu - ie^3 \int \frac{dk}{8\pi^2} \frac{-2}{4} \frac{k^5}{k^6} \gamma_\alpha \gamma_\mu \gamma^\alpha \tag{4.36}$$

$$= -ie\gamma_\mu - \frac{ie^3}{8\pi^2} \gamma_\mu \ln \Lambda \tag{4.37}$$

$$Z^{-1}_{T_{\bar{\psi}\psi A}} = 1 + \frac{e^2}{8\pi^2} \ln \frac{\Lambda}{\mu} \tag{4.38}$$

$$Z_{T_{\bar{\psi}\psi A}} = 1 - \frac{e^2}{8\pi^2} \ln \frac{\Lambda}{\mu} \tag{4.39}$$

Check gauge-invariance of $m_R(\mu)$:

Figure 4.7: Added Stuff.

$$added\ stuff \propto \int^\Lambda \frac{d^4k}{(2\pi)^4} \frac{\alpha k_\mu k_\nu}{k^2} \frac{\gamma_\mu (\not{p} - \not{k} + m) \gamma_\nu}{k^2 (p-k)^2} \tag{4.40}$$

$$need \propto \not{p} - m \tag{4.41}$$

$$\log : 2a \frac{1}{8\pi^2} \int^\Lambda \frac{dk k^3}{k^6} (\frac{1}{4}(-2)\not{p} + \frac{1}{4} 4m) \tag{4.42}$$

$$linear : \frac{1}{8\pi^2} \int^\Lambda \frac{d\Omega dk k_\mu k_\nu (-)(\gamma_\mu \not{k} \gamma_\nu (2pk) k^3)}{k^8} \tag{4.43}$$

$$= \frac{1}{8\pi^2} \int^\Lambda \frac{d\Omega dk (-) \not{k} (2pk) k^3)}{k^6} \tag{4.44}$$

$$= \frac{1}{8\pi^2} \int^\Lambda dk \frac{\frac{1}{4}(-2)\not{p} k^3}{k^4} - \not{p} + m \tag{4.45}$$

Effective mass: $m'_R(\mu_1)$ equivalent to $m^2_R(\mu_2)$ if same m_0, Λ

$$\frac{m_R'(\mu_1)}{1-3\frac{e^2}{8\pi^2}\ln\frac{\Lambda}{\mu_1}} = \frac{m_R^2(\mu_2)}{1-3\frac{e^2}{8\pi^2}\ln\frac{\Lambda}{\mu_2}} \tag{4.46}$$

$$\frac{m_R^2(\mu_2)}{m_R'(\mu_1)} = \frac{1-3\frac{e^2}{8\pi^2}\ln\frac{\Lambda}{\mu_1}}{1-3\frac{e^2}{8\pi^2}\ln\frac{\Lambda}{\mu_2}} \tag{4.47}$$

$$\stackrel{!}{=} 1-3\frac{e^2}{8\pi^2}\ln\frac{\mu_2}{\mu_1} \tag{4.48}$$

$$\frac{m_R(\mu)}{\partial\ln\mu} = 3\frac{e^2}{8\pi^2} \tag{4.49}$$

Renormalization group: N.B. : $e^2\mu$ too.
Physical significance:

1. With normalization at μ, no large loop for processes with parameter $a \sim \mu$.
2. Internally, not rotate for μ to Ω.

Anticipate:

$$\frac{dg(\mu)}{d\ln\mu} = -b_0 g^3 \ (b_0 > 0) \tag{4.50}$$

$$\frac{d\frac{1}{g^2}}{dt} = 2t_0 \tag{4.51}$$

$$\frac{1}{g^2(\mu)} - \frac{1}{g^2(\mu_0)} = 2b_0\ln\frac{\mu}{\mu_0} \tag{4.52}$$

$$e^{\frac{1}{2b_0 g^2(\mu)}} = e^{\frac{1}{2b_0 g^2(\mu_0)}}\frac{\mu}{\mu_0} \tag{4.53}$$

$$\mu = [\mu_0 e^{\frac{1}{2b_0 g^2(\mu_0)}}]e^{\frac{1}{2b_0 g^2(\mu)}} \tag{4.54}$$

Take $\mu \to \infty$, dimensional transmutation perturbative coupling.
Lattice: Given $\frac{1}{g^2(\mu)}$, determine $\mu \propto \frac{1}{a}$. Measure $\frac{M}{\mu}$ fixed, independent of μ as $g \to 0$.
Good news: Check approach to continuous limit. Bad news: Decreasing $g(\mu)$ required exponentially bigger lattice.
With opposite sign for b_0, coupling blows up as $\mu \to \infty (a \to 0)$, no limiting theory guaranteed.

Running coupling and dimensional transmutation:

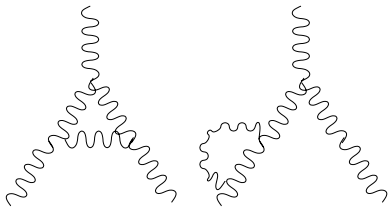

Figure 4.8: Running Coupling

$$\frac{dg(\mu)}{d\ln\mu} = -b_0 g^3 \quad (b_0 > 0) \tag{4.55}$$

$$\frac{d\frac{1}{g^2}}{d\ln\mu} = 2b_0 \tag{4.56}$$

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + 2b_0 \ln\frac{\mu}{\mu_0} \tag{4.57}$$

Bare charge $\to 0$

$$\frac{\mu}{\mu_0} = e^{\frac{1}{2b_0 g^2(\mu)}} e^{\frac{1}{2b_0 g^2(\mu_0)}} \tag{4.58}$$

$$\mu_0 = e^{\frac{1}{2b_0 g^2(\mu)}} \mu e^{\frac{1}{2b_0 g^2(\mu_0)}} \tag{4.59}$$

Identify from $\mu \to \infty$ limit [finite].
More accurate:

$$\frac{dg}{d\ln\mu} = -b_0 g^3 + b_1 g^5 \tag{4.60}$$

$$\frac{d\frac{1}{g^2}}{d\ln\mu} = 2b_0 - 2b_1 \frac{1}{\frac{1}{g^2}} \tag{4.61}$$

$$\frac{du}{dt} = 2b_0 - \frac{2b_1}{u} \tag{4.62}$$

$$u_0 = 2b_0 t \tag{4.63}$$

$$\frac{du}{dt} = 2b_0 - \frac{b_1}{b_0 t} \tag{4.64}$$

$$u \simeq 2b_0 t - \frac{b_1}{b_0}\ln t \tag{4.65}$$

$$\frac{1}{g^2(\mu)} \simeq 2b_0 \ln\frac{\mu}{\mu_0} - \frac{b_1}{b_0}\ln\ln\frac{\mu}{\mu_0} \tag{4.66}$$

$$\simeq 2b_0 \ln\frac{\mu}{\mu_0} - \frac{b_1}{b_0}\ln\frac{1}{2b_0 g^2} \tag{4.67}$$

$$\ln\mu_0 = \ln\mu - \frac{1}{2b_0 g^2} + \frac{b_1}{2b_0^2}\ln\frac{1}{2b_0 g^2} \tag{4.68}$$

$$\mu_0 \simeq \mu e^{-\frac{1}{2b_0 g^2(\mu)}}[\frac{1}{g^2(\mu)}]^{\frac{b_1}{2b_0^2}} \; finite\ as\ \mu\to\infty \tag{4.69}$$

Relation to lattice (fundamental): Hold μ_0 fixed (reference mass)

$$a(g) \sim \frac{1}{\mu} \sim e^{-\frac{1}{2b_0 g^2}}[\frac{1}{g^2}]^{\frac{b_1}{2b_0^2}}\frac{1}{\mu_0} \tag{4.70}$$

Small a corresponds to small g. N.B.: Exponential dependence. Test: Physical masses stay constant as $g \to 0$. Explicitly: Measures correlation fall-off $e^{-\lambda n}$, where n is the lattice sparcing.

This is e^{-ML} with $L = an$ for $M = \frac{\lambda}{a}$.

$$e^{-\lambda n} = e^{-\frac{\lambda}{a}an} \tag{4.71}$$

$$\to \mu_0 e^{\frac{1}{2b_0 g^2}[\frac{1}{g^2}]^{\frac{b_1}{2b_0^2}}\lambda} \tag{4.72}$$

Fixing μ_0, we see $\lambda \to 0$ as $g \to 0$. Large correlation lengths, variating of lattice artifacts. Alternatively, $aM \to 0$ without asymptotic freedom we would be lost here, in tertudation theory. Long correlation lengths can occur at critical points. To nail things down let's look at this more closely:

$$\frac{dg}{d\ln\mu} \cong -b_0(g - g_C) \tag{4.73}$$

$$\ln(g - g_C) = -b_0 \ln\mu \tag{4.74}$$

$$\frac{g(\mu) - g_C}{g(\mu_0) - g_C} = (\frac{\mu_0}{\mu})^{b_0} \tag{4.75}$$

$$\mu = \mu_0(\frac{g(\mu) - g_C}{g(\mu_0) - g_C})^{\frac{1}{b_0}} \tag{4.76}$$

so we have, $g(\mu) \to g_C$

$$e^{-n\lambda} = e^{-\frac{\lambda}{a}na} \tag{4.77}$$

$$M = \frac{\lambda}{a} \tag{4.78}$$

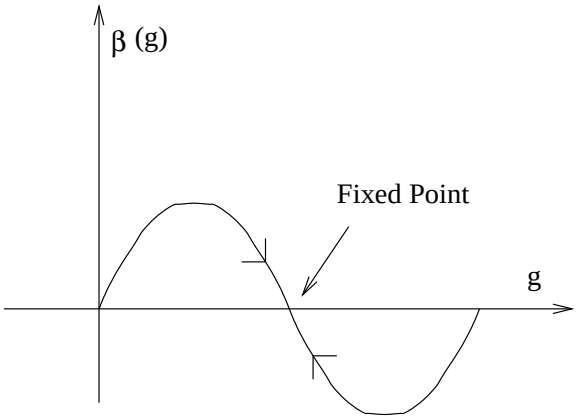

Figure 4.9: Fixed Point

$$a(g(\mu)) \sim \frac{1}{\mu} \sim \frac{1}{\mu_0}(\frac{g(\mu) - g_C}{g(\mu_0) - g_C})^{\frac{1}{b_0}} \tag{4.79}$$

$M = \frac{\lambda}{a}$ fixed $\Rightarrow \lambda \to 0, \frac{1}{M} \gg a$. Alternatively, IR fixed points, but these do not support non-zero masses.

Antiscreening as paramagnetism:

Vacuum polarization:

$$E.D \propto \epsilon F_{oi} F^{oi} \tag{4.80}$$
$$B.H \propto \mu^{-1} F_{ij} F^{ij} \tag{4.81}$$

In reality

$$\epsilon\mu = 1 \tag{4.82}$$

either is varying g.

Antiscreening ($\epsilon < 1$) = paramagnetism ($\mu > 1$) magnetism is easier to think about, since particles do not get produced.

$$E_{class.} = \frac{1}{2g^2} B^2 \tag{4.83}$$

But zero point energies, $\frac{1}{2}h\omega$ (Bose), $-\frac{1}{2}h\omega$ (Fermi) would like to throw it out, but it is field-dependent and cut off dependent.

Interpretation using running coupling:

Wilson approach: Contribution of modes between $\mu_0 + \mu$ (say $\mu > \mu_0$) is

$$= -n \ln \frac{\mu}{\mu_0} B^2 \tag{4.84}$$

$$E = \underbrace{\frac{1}{2g^2(\mu_0)}}_{cut\ off\ at\ \mu_0} B^2 \tag{4.85}$$

$$= \underbrace{(\frac{1}{2g^2(\mu)} - n \ln \frac{\mu}{\mu_0})}_{cut\ off\ at\ \mu} B^2 \tag{4.86}$$

$$n = b_0 \tag{4.87}$$

Result and physical interpretation:

$$n = \frac{1}{96\pi^2}[-\{T(R_0) - 2T(R_{\frac{1}{2}} + 2T(R_1))\}] + \frac{1}{96\pi^2}[3\{-2T(R_{\frac{1}{2}} + 8T(R_1))\}] \tag{4.88}$$

To aid in physical interpretation, pick a particular direction (e.g., τ_3) in internal space for B, and pretend it is electromagnetism.

First term from orbital diamagnetism (Lambda). Note 2 for polarizations, $-$ for fermion.

N.B. This applies to 0–point energy of virtual particles. Real particles do not have this.

Second term from spin paramagnetism (Pauli). Note g–factor 2 (also for vectors), $-$ for fermion, $\frac{3}{1}$ ratio as in solid state.

$$\frac{S^2}{S^2_{\frac{1}{2}}} = 4 \tag{4.89}$$

AF comes from gluon paramagnetism. Reference: F. W., RMP 71 S85 (Ceturean Issue).

Learning from QED: With these insights, we could extract the non-Abelian β function from the Abelian one.

Numerical value:

$$QCD: \ T(R_{\frac{1}{2}}) = f\frac{1}{2} \tag{4.90}$$

$$T(R_1) = 3 \tag{4.91}$$

$$b_0 = \frac{1}{96\pi^2}[+2\frac{f}{2} - 2 \times 3 + 3(-2\frac{f}{2} + 8 \times 3)]$$

$$= \frac{1}{96\pi^2}[66 - 4f] = \frac{1}{16\pi^2}[11 - \frac{2}{3}f] \tag{4.92}$$

$b_0 > 0$ for $f \leq 16$. Gluons heavily dominate for $f = 3$.
Application for unified theories:

$$SU(3) \qquad -8F + 66 \tag{4.93}$$

$$SU(2) \qquad F(\underbrace{-4}_{fermion\ factor} \times \underbrace{4}_{number\ of\ doublets} \times \underbrace{\frac{1}{2}}_{T_{doublet}}) + 44$$

$$= \quad -8F + 44\,\frac{1}{2} \quad Higgs \tag{4.94}$$

$$U(1) \qquad normalized\ charge : \begin{pmatrix} 2 & & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & -3 & & \\ & & & & -3 & \end{pmatrix} \frac{1}{\sqrt{60}} \tag{4.95}$$

$$\sum q^2 \quad = \quad F(\frac{1}{2} + \frac{1}{60}(3 \times 4^2 + 6(-1)^2 + 1(-6)^2)) \tag{4.96}$$

$$= \quad 2F \overset{W.S.B.}{\longrightarrow} -8F \; - \frac{3}{10} \quad Higgs \tag{4.97}$$

3 equations: constraint, t_u, g_u

$$\frac{1}{g_u^2} \quad = \quad b^i t_u + \frac{1}{g_i^2} \tag{4.98}$$

$$\frac{1}{g_2^2} - \frac{1}{g_3^2} \quad = \quad (b^3 - b^2) t_u \tag{4.99}$$

Note: unchanged by complete multiplets (no F)

$$\frac{\frac{1}{g_2^2} - \frac{1}{g_3^2}}{\frac{1}{g_1^2} - \frac{1}{g_2^2}} = \frac{b^3 - b^2}{b^2 - b^1} \tag{4.100}$$

Fixing scale: note exponential dependence

$$t_u \quad = \quad \frac{\frac{1}{g_2^2} - \frac{1}{g_3^2}}{b^3 - b^2} = \frac{1}{b^3 g_u^2} - \frac{1}{b^3 g_3^2} = \frac{1}{b^2 g_u^2} - \frac{1}{b^2 g_2^2} \tag{4.101}$$

$$\frac{1}{g_u^2} \quad = \quad \frac{\frac{1}{b^2 g_2^2} - \frac{1}{b^3 g_3^2}}{\frac{1}{b^2} - \frac{1}{b^3}} = \frac{b^3 \frac{1}{g_2^2} - b^2 \frac{1}{g_3^2}}{b^3 - b^2} = \frac{1}{2}(\frac{1}{g_2^2} + \frac{1}{g_3^2}) + \frac{1}{2}\frac{b^3 + b^2}{b^3 - b^2}(\frac{1}{g_2^2} - \frac{1}{g_3^2}) \tag{4.102}$$

4.1 SUSY Modifications

1. Gauge Modification

$$2R_{\frac{1}{2}} - 2R_1 \tag{4.103}$$
$$-6R_{\frac{1}{2}} - 24R_1 \tag{4.104}$$
$$R_{\frac{1}{2}} = \underbrace{11}_{group\ theory} \times \underbrace{2}_{chialitic} \times \underbrace{\frac{1}{2}}_{majorna} \Rightarrow factor\frac{9}{11} \tag{4.105}$$
$$66 \rightarrow 54 \tag{4.106}$$
$$44 \rightarrow 36 \tag{4.107}$$

2. Higgs Modification

$$2\ doublets \tag{4.108}$$
$$fermion\ partner \tag{4.109}$$
$$-1 \rightarrow 2 \times (-1 + \underbrace{2-6}_{\times\frac{1}{2}})\ like\ 6\ Higgs\ doublet \tag{4.110}$$

3. Matter Modification

$$\times\frac{3}{2}\ by\ same\ reasoning \tag{4.111}$$

Example: $N = 4$ gauge field. 4 adjoint fermions. 6 Higgs.

$$\begin{array}{rcc} diamagnetism: & 2 & -8 \\ paramagnetism: & 8 & -8 \end{array} \tag{4.112}$$

Numerical aspects:

$$\frac{b^3 - b^2}{b^2 - b_1} = \frac{22 + \frac{1}{2}}{44 - \frac{1}{2} + \frac{3}{10}} = \frac{75}{146} \simeq 0.514 \tag{4.113}$$

SUSY:

$$\frac{b^3 - b^2}{b^2 - b_1} = \frac{18 + \frac{6}{2}}{36 + 6(-\frac{1}{2} + \frac{3}{10})} = \frac{35}{58} \simeq 0.604 \tag{4.114}$$

$\frac{1}{b^3 - b^2}$:

$$\frac{1}{22+\frac{1}{2}} \rightarrow \frac{1}{18+\frac{6}{2}} \tag{4.115}$$

$$0.044 \rightarrow 0.048 \tag{4.116}$$

Raising scale looks small but appears in exponential (p–decay).

$$\frac{b^3+b^2}{b^3-b^2} = \frac{100+16F-\frac{1}{2}}{22+\frac{1}{2}} \overset{F=3}{=} \frac{62-\frac{1}{2}}{22+\frac{1}{2}} = \frac{41}{15} \simeq 2.73 \tag{4.117}$$

SUSY:

$$\frac{b^3+b^2}{b^3-b^2} = \frac{90+24F-3}{22+3} \overset{F=3}{=} \frac{3}{5} \simeq 0.60 \tag{4.118}$$

b^2: (very little remaining)

$$44-8F-\frac{1}{2} = 19\frac{1}{2} \rightarrow 36-12F-3 = -3 \tag{4.119}$$

Other applications:

$$\frac{m_b}{m_\tau} \rightarrow 3 \; from \; minimal \; SU(5) \tag{4.120}$$

Vacuum instability and Colman–Weinberg:

$$\varphi^4 \lambda(\frac{\varphi}{\mu}) \tag{4.121}$$

interpret as energy from extra nodes.
Too many fermions. moment μ, field strength φ (as in our magnetic calculation).

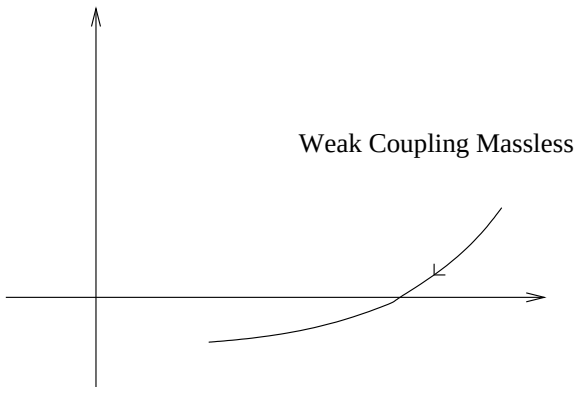

Figure 4.10: Weak Coupling Massless

Table 4.1: Running of coupling

	SM	$MSSM$	
$SU(3)_{b_3^3}$	$66-8F$	$54-12F$	
$SU(2)_{b_3^2}$	$44-8F-\frac{1}{2}$	$36-12F-3$	nearly flat
$U(1)_{b_0^1}$	$-8F-\frac{3}{10}$	$-12F-\frac{9}{5}$	
$\frac{\frac{1}{g_2^2}-\frac{1}{g_3^2}}{\frac{1}{g_1^2}-\frac{1}{g_2^2}}=\frac{b^3-b^2}{b^2-b^1}$	0.514	0.604	agrees with expectation
$t_u=\frac{\frac{1}{g_2^2}-\frac{1}{g_3^2}}{b^3-b^2}$	0.044	0.048	larger scale important for p-decay close to Planck scale
$\frac{1}{g_u^2}=\frac{\frac{1}{b^2g_2^2}-\frac{1}{b^3g_3^2}}{\frac{1}{b^2}-\frac{1}{b^3}}$	2.73	0.60	families and extra stuff appear here bigger coupling desirable?

Running of coupling – summary sheet

Other applications: $N=4$ SUSY, glue, 4 Majorna fermions, 6 scalars. Dia: count DOF. Para: 4 fermion, $\beta=0$

Normal SU(5):

$$\frac{m_b}{m_\tau}=1\ at\ unification \tag{4.122}$$

Normalizes to

$$\frac{m_b}{m_\tau}\simeq 3 \tag{4.123}$$

Vacuum instability

$$\lambda(\frac{\mu}{\mu_0})\varphi^4\ negates\ (AF)\ contribution \rightarrow \lambda(\frac{\varphi}{\mu_0})\varphi^4\ from\ fermions \tag{4.124}$$

Variational interpretation?

$$<\varphi>\leftarrow a_k^+, a_k \tag{4.125}$$

mix mode in $|k|$–dependent manner. C.F. our magnetic field calculation.

Colman-Weinberg:

Electroweak breaking through leavry sTop. t instability ulicrod when $\hat{t}$ comes in.

PQCD:

Simplest case: $e^+e^-\rightarrow$ hadrons

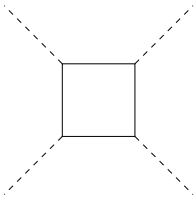

Figure 4.11: Vacuum Instability

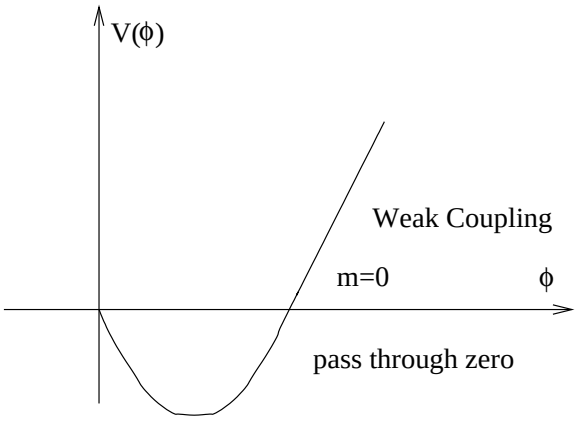

Figure 4.12: Spontanous Symmetry Breaking

$$\frac{\sigma(p, g^{(\mu)}, \mu)}{\sigma(\lambda p, g^{(\mu)}, \mu)} = \sigma(\lambda p, g(\lambda\mu), \lambda\mu) \tag{4.126}$$

$$= \frac{1}{\lambda^2} \underbrace{\sigma(p, g(\lambda\mu), \mu)}_{expand\ perturbutively} \tag{4.127}$$

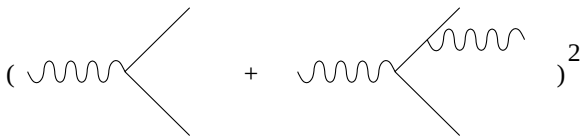

Figure 4.13: Expand Perturbatively.

Potential problem from soft and collisions divergence. They do not occur in inclusive qualities.

Jet phenomena: Observation, heuristic explanation, hard scattering, rate. Sterman-Weinberg comes, energy bite. Stiull no scale – same argument. Extremum pattern. Hadmic gets over 6 nodes of reguitode.

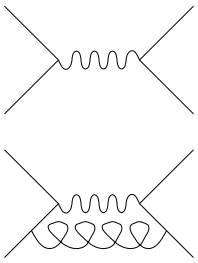

Figure 4.14: W–Exchange.

OPE, DIS: W–exchange.
Nodes up to $k^2 \sim m_\omega^2$.
Simplify to

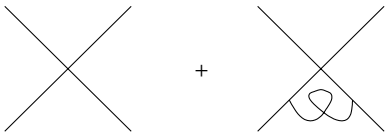

Figure 4.15: Simplify.

Operator versions:

$$J(x)J(\frac{X}{2}) \stackrel{small\ x}{\longrightarrow} \sum C_i(x)O_i(o) \tag{4.128}$$

Singularities of C ordered by dimensions of O_i keep low dimensions. $C_i(x, g, \mu)$ includes renormalization of O.

$$C_i(x, g, \mu) = [\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_\theta(\lambda)]C_i = 0 \tag{4.129}$$

$$\begin{aligned} C_i(\lambda^{-1}x, g(\mu_0), \mu_0) &= C_i(x, g(\lambda\mu_0))e^{-\int_{g_0}^{g_1}\gamma_\theta\frac{dg}{\beta}} \\ &= C_i(x, g(\lambda\mu_0))(\frac{\bar{g}}{g})^{-\frac{c_0}{b_0}} \end{aligned} \tag{4.130}$$

One can also have operators mixing. For DIS, light–cone singularities (tracing quarks and glouns) are important.

$$J^\mu(\frac{x}{2})J^\nu(-\frac{x}{2}) \stackrel{x^2\to 0}{\approx} \sum C_{(x)}^{(i)}O^{\mu\nu\alpha_1\cdots\alpha_n}(0)x_{\alpha_1\cdots x_{\alpha_n}} \tag{4.131}$$

but not individual component.

Twist $\equiv$ dimension – spin

$$< O > \rightarrow p^\mu p^\nu \cdots \ (symmetric) \tag{4.132}$$

$$x \rightarrow \frac{\partial}{\partial g^2} \ or \ \frac{g}{g^2} \tag{4.133}$$

$(\frac{pq}{q^2})^2$ is what appears.
B_j limit:

$$g^2 \rightarrow \infty \tag{4.134}$$

$\frac{2pq}{q^2}$ finite ($\equiv \frac{1}{x}$).
QCD has twist–tor families of operators (respectively chirality).

$$\overline{g}_{(L)} \gamma_{\alpha_1} \overleftrightarrow{\nabla}_{\alpha_2} \cdots \overleftrightarrow{\nabla}_{\alpha_n} q_{(L)} \qquad D = 3 + n - 1, J = n \tag{4.135}$$

$$G^a_{\mu\alpha_1} \overleftrightarrow{\nabla}_{\alpha_2} \cdots \overleftrightarrow{\nabla}_{\alpha_{n-1}} G^a_{\alpha_n \mu} \qquad D = 4 + n - 2, J = n \tag{4.136}$$

First family includes nonsinglets, currents. $T_{\mu\nu}$ also appears (but not by itself). Twist J governs J^{th} moment of appropriate structure function.

Many testable predictions: Parton model sum rules and g^2 correlations. CG relation and g^2 correlations. E: Max with o^{th} order correlations. B_j scalar deviations.

Chapter 5

Anomalies

When classical symmetry can not be maintained in QFT, we say we have an anomaly. Note, this is quite different from spontenous symmetry breaking.

We are just discussed one type, anomaly in scale invariance. Mass without mass (dimensional transmutation). PQCD unification scale/renormalization.

Another kind, with a rather difficult flavor is connected with chiral symmetry. Many applications:

- Eliminate extra $U_A(1)$ symmetry of massless QCD (approximate for real QCD, but still too good).
- Eliminate I_3^S (axial isospin) of QCD $\times$ QCD $\Rightarrow m_{n'} >> m_\pi$ (actually, modify it) $\pi^\circ \to 2\gamma$
- Constraint on what QFTs are consistent by demanding no anomalies in gauge symmetries.
- Connections with topology/solitons.
- $\cdots$
- Hawking radiation (recent work).

Begin with a very concrete low basis approach. $\overset{A}{\Delta}\ V\ V$ graph for mass fermions.

$$I^{\lambda\mu\nu} \equiv i \int \frac{d^4p}{(2\pi)^4} tr(\gamma^\lambda \gamma^5 \frac{1}{\not{p} - \not{q}} \gamma^\nu \frac{1}{\not{p} - \not{k}_1} \gamma^\mu \frac{1}{\not{p}} + crossed) \tag{5.1}$$

Check if $k_1 \mu I^{\lambda\mu\nu} \overset{?}{=} 0$

$$\frac{1}{\not{p} - \not{k}_1} \not{k}_1 \frac{1}{\not{p}} \quad = \quad \frac{1}{\not{p} - \not{k}_1} - \frac{1}{\not{p}} \tag{5.2}$$

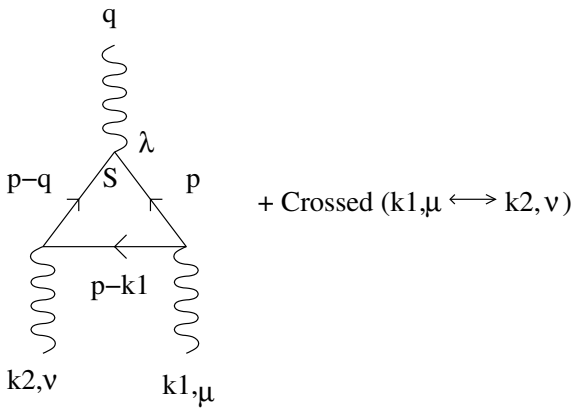

Figure 5.1: Low Basis.

$$\frac{1}{\not p - \not q}(\not q - \not k_2)\frac{1}{\not p - \not k_2} = \frac{1}{\not p - \not q} - \frac{1}{\not p - \not k_2} \tag{5.3}$$

$$k_{1\mu} I^{\lambda\mu\nu} = i\int \frac{d^4p}{(2\pi)^4} tr[\gamma^\lambda\gamma^5(\frac{1}{\not p - \not q}\gamma^\nu\frac{1}{\not p - \not k_1} - \frac{1}{\not p - \not k_2}\gamma^\nu\frac{1}{\not p})] \tag{5.4}$$

This is superficially linearly divergent (note $\sum pp \to 0$ in numerator). Formally, $p \to p - k_1$ in 2^{nd} term makes them cancel. But, linear divergence is dangerous – shifts can leave finite surface terms at $|p| \to \infty$.

Careful shift:

$$f(p+a) \approx \underbrace{f(p)}_{cancels} + \underbrace{a^\mu\frac{\partial}{\partial p^\mu}f(p)}_{finite\ \int} + \underbrace{highers}_{\to\infty} \tag{5.5}$$

$$\int d^4p[f(p+a) - f(p)] = \lim_{p\to\infty}\int \underbrace{d\Omega}_{angular\ average}\frac{p_\mu}{p}f(p)2\pi^2p^3\underbrace{ia^\mu}_{from\ euclidean\ rotation} \tag{5.6}$$

$$f(p) = tr\gamma^\lambda\gamma^5\frac{1}{p - k_2}\gamma^\nu\frac{1}{p} = \frac{4i\epsilon^{\tau\nu\sigma\lambda}k_{2\tau}p_\sigma}{(p-k_2)^2p^2} \tag{5.7}$$

$$a^\mu = k_1^\mu \tag{5.8}$$

$$k_{1\mu}I^{\lambda\mu\nu} = \lim_{p\to\infty} d\Omega\frac{4i\epsilon^{\tau\nu\sigma\lambda}k_{2\tau}(p_\sigma p_{\mu i})}{p^4(2\pi)^2}2\pi^2p^3ik_1^\mu$$

$$= \frac{i}{8\pi^2}\epsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}\ (\neq 0) \tag{5.9}$$

However the whole calculation is bogous because we could shift by anything in internal momentum.

$$I^{\lambda\mu\nu}(p \to p+a, k_1, k_2) - I^{\lambda\mu\nu}(k_1, k_2) = \frac{i}{8\pi^2}\epsilon^{\sigma\nu\mu\lambda}a_\sigma + crossed \qquad (5.10)$$

e.g., choosing $a_\sigma = \frac{1}{2}(k_2 - k_1)$ gives conceived V, (or CVC). This corresponds to symmetric integration.

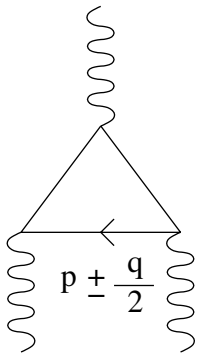

Figure 5.2: Symmetric Integration.

With this choice, though, the naive axial divergence gets doubled, just cancelled.

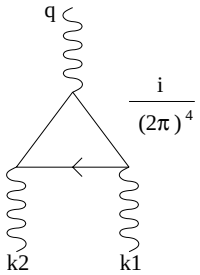

Figure 5.3: Axial Divergence.

$$\frac{1}{\not p}\not q\gamma_5\frac{1}{\not p - \not q} = \frac{1}{\not p}(\not p - (\not p - \not q))\gamma_5\frac{1}{\not p - \not q} = \gamma_5\frac{1}{\not p - \not q} + \frac{1}{\not p}\gamma_5 \qquad (5.11)$$

As before, but with $\lambda \leftrightarrow \mu$

$$\begin{aligned}\frac{i}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}tr\gamma_5\frac{1}{p-q}\gamma_\nu\frac{1}{p-k_1}\gamma_\mu &= k_{1\mu}[\epsilon^{\mu\nu\lambda\sigma}k_{2\sigma}] \\ &= -\underbrace{(-(k_1+k_2)_\lambda)}_{q_\lambda}[\epsilon^{\mu\nu\lambda\sigma}k_{2\sigma}] \qquad (5.12)\end{aligned}$$

This leads to

$$\partial_\mu J^{\mu S} = \frac{e^2}{(4\pi)^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{5.13}$$

2 fields, 4 terms $\Rightarrow$ factor γ.

Not a very satisfactory derivation, of course. Much better is to use Pauli-Villars regulator, add massive spin $\frac{1}{2}$ boson for opposite sign in loop. Take $M \to \infty$ at the end, because we do not want this in the physical spectrum.

Now we can shift with a clean conscience

$$tr\gamma^\lambda\gamma_5\frac{1}{\not p - \not q - M}\gamma^\nu\frac{1}{\not p - \not k_1 - M}\gamma^\mu\frac{1}{\not p - M} \tag{5.14}$$

where $\not k_1 = (\not p - M) - (\not p - \not k_1 - M)$, vector is no problem.

Axial vector

$$\not q = (\not p - M) - (\not p - \not q - M) \tag{5.15}$$

$$\begin{aligned}\frac{1}{\not p - M}\not q\gamma_5\frac{1}{\not p - \not q - M} &= \gamma_5\frac{1}{\not p - \not q - M} + \frac{1}{\not p - M}\gamma_5\frac{\not p - \not q + M}{\not p - \not q - M} \\ &= odd\ term + 2M\frac{1}{\not p - M}\gamma_5\frac{1}{\not p + \not q - M}\end{aligned} \tag{5.16}$$

$$\begin{aligned}answer &\propto M\int d^4p tr\gamma^5\frac{1}{\not p - \not q - M}\gamma^\nu\frac{1}{\not p - \not k_1 - M}\gamma^\mu\frac{1}{\not p - M} \\ sim &\ \frac{M\epsilon^{\mu\nu\lambda\sigma}(p_{\to 0}, k_1, k_2)}{(p^2 - M^2)^3}\ (finite\ as\ M \to \infty)\end{aligned} \tag{5.17}$$

This gives the same final answer, of course.

Chapter 6

Conversion of Anomalies

Scholium:

1. Other regular scheme: Dimensional regularization: subtlety in continuing ϵ symbol. Lattice regularization: doubling of fermions. The anomaly can be derived from axions (CVC, bose statistics, ...) so they all must give the same answer.

2. Higher orders do not renormalize the anomaly. Looking in one particles gives. consequence factor, allows shifts (Adler-Bardeen).

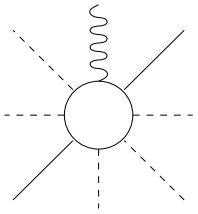

Figure 6.1: Higher Orders.

   This correlates with

$$e^2 F\tilde{F} + f^2 \to F\tilde{F} + \frac{1}{e^2}F^2 \tag{6.1}$$

   numerical coefficient.

3. Closely related in non-decoupling of heavy quarks in $h \to gg$ vertex.

$$\int d^a p \frac{trkkM}{(p^2 + M^2)^3} \sim \frac{1}{M} \tag{6.2}$$

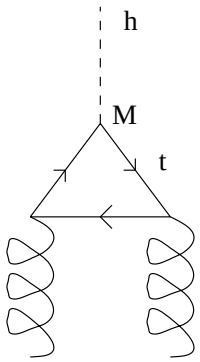

Figure 6.2: Non-Decoupling.

finite as $M \to \infty$

Key to phenomenology.

4. This is also closely commuted to trace anomaly, scaling anomaly. Note these do get renormalized.

5. In path $\int$, the anomaly arises from non-invariance of the measure (Fujikawa) .

6.
$$\partial_\mu j^{\mu 5} = K\epsilon^{\alpha\beta\sigma\delta}\partial_\alpha A_\beta \partial_\gamma A_\delta = K\partial_\alpha(\epsilon^{\alpha\beta\sigma\delta}A_\beta\partial_\gamma A_\delta) \tag{6.3}$$

so there is a conserved current

$$\tilde{j}^{\mu S} = j^{\mu S} - K\epsilon^{\alpha\beta\sigma\delta}A_\beta\partial_\gamma A_\delta \tag{6.4}$$
$$\partial_\mu \tilde{j}^{\mu S} = 0 \tag{6.5}$$

This is not gauge-invariant, however, only j^μ clearly current to physical states.

7. Anomaly cancellation in the standard model

$$SU(3)_V \times SU(2)_L \times U(1)_{complicated} \tag{6.6}$$
$$SU(2)^3 \quad (vanishing\ tr(\tau\{\tau\ \tau\})) \tag{6.7}$$
$$SU(2)^2 \times U(1) \tag{6.8}$$
$$\underbrace{\frac{1}{6}}_{quarks} \cdot \underbrace{3}_{leptors} - \frac{1}{2} = 0 \tag{6.9}$$

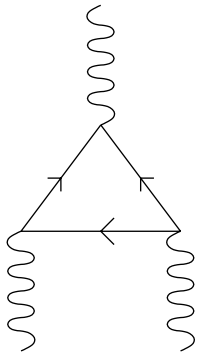

Figure 6.3: L–Handed.

note connection.
$U(1)^3$:

$$L \;:\; (\frac{1}{6})^3 \cdot 6 + (-\frac{1}{2})^3 \cdot 2 = -\frac{2}{9} \tag{6.10}$$

$$R \;:\; (\frac{2}{3})^3 \cdot 3 + (-\frac{1}{3})^3 \cdot 3 + (-1)^3 = -\frac{2}{9} \tag{6.11}$$

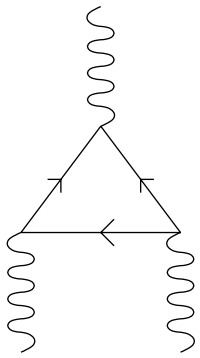

Figure 6.4: $U(1)^3$.

8. Non-cancellation of B, L anomalous. All L–handed $(\frac{1}{2})^2 + (-\frac{1}{2})^2$. $B - L$ is not anomalies.

9. Anomalies relevant to QCD.

$$m_u, m_d \approx 0 \tag{6.12}$$

QCD + QED

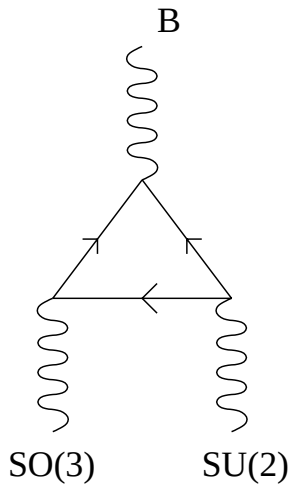

Figure 6.5: Non-Cancellation.

A meta-spealation I find fascinating: Symmetry restoration by anomalies. This would in some cases give symmetry $\Rightarrow$ QM.

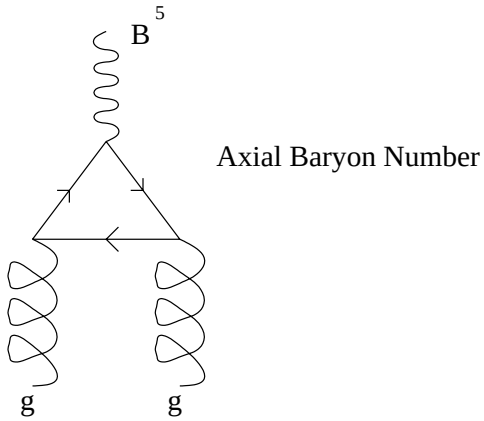

Figure 6.6: QCD.

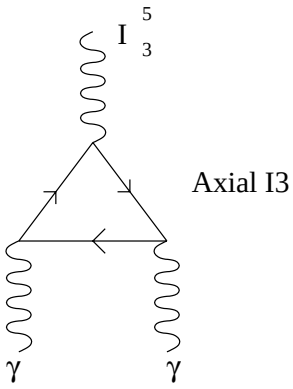

Figure 6.7: QCD + QED.

Chapter 7

Chiral Symmetry

Chiral symmetry in the strong interaction (and specifically in QCD). Exploiting an approximate, hidden symmetry to simply description of π_i's and their interaction weak processes involving hadrons (recently) some modern processes.

Leading ideas (predicting QCD) :

1. There is a good approximate symmetry of the strong interaction under algebra $SU(2)_L \times SU(2)_R$ (or $SU(3)_L \times SU(3)_R$) with $SU(2)_{L+R} = isospin$. Small instincs breaking.

2. This symmetry is spontaneously violated in the ground state. Pseudoscalar mesons (π, κ, ν) are collective modes (Nambu-Goldstone bosons) associated with broken symmetry direction.

3. The generators of these symmetries appear in the electroweak interactions. In the standard model, these hypotheses are consequences of

$$m_u, m_d \ll \Lambda_{QCD} \tag{7.1}$$

$$m_s \leq \Lambda_{QCD} \tag{7.2}$$

$$< \bar{u}u > = < \bar{d}d > = < \bar{s}s > \neq 0 \tag{7.3}$$

in massless limit.

N.B. : It is very important that turning of the masses is a soft perturbation so that we can do perturbation theory around the massless limit.

$$L_{QCD} = L_{m=0} + m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s \tag{7.4}$$

Note: tricky PT due to massless particles.

The specific realization in QCD is more powerful:

(a) Link to PQCD – corrections

(b) Concrete realization of breaking

i. L_1 transforms as $(3,\bar{3})+(\bar{3},3)$ under $SU(3)_L \times SU(3)_R$

ii. Contributions from anomalies as mentions above.

7.1 History and Sketch of Example Application

Goldberger-Treiman formula (1957)

$$g_{\pi NN} = \frac{g_A M_N}{f_\pi} \tag{7.5}$$

This works well, but their derivation was cheesy.

Nambu (1960-62) relate to approximate symmetry and correlate with lightness of π mesons.

$$\begin{aligned}
< O|j^s_\mu|\pi > &\sim f_\pi p_\mu \ (measured\ in\ \pi \to \mu\nu) && (7.6)\\
< O|\partial j|\pi > &\sim f_\pi p^2 = f_\pi m_\pi^2 \approx 0 && (7.7)\\
< n|j^s_\mu|\pi > &= g_A \bar{u}(n)\gamma_1\gamma_\mu u(p)\ (+P.S.) && (7.8)\\
0 &= < n|j^s_\mu|\pi > && (7.9)\\
&= \underbrace{g_A M_N}_{direct} - \underbrace{f_\pi g_{\pi NN}}_{\pi\ pole} && (7.10)
\end{aligned}$$

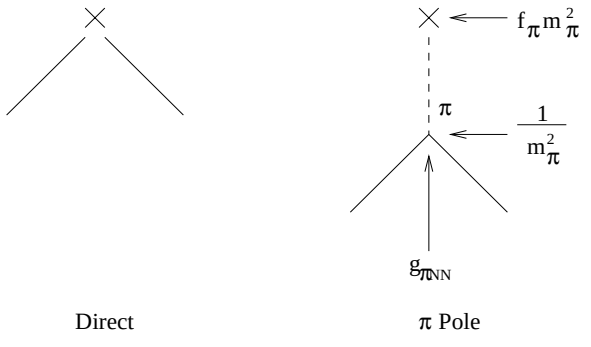

Figure 7.1: Goldberger-Treiman.

$\Rightarrow$ Goldberger-Treiman.

This is the tip of an iceberg of applications, as alluded to above.

7.2 Meson Masses, $U_A(1)$ Problem

Standard (Gellmann – Oakes – Ronne) GM-O-R → (Gellmann – Okulbo) GM-O will discuss this using effective Lagrangian with

$$< \bar{g}_{Li}, g_R^j >= \nu \delta_i^j \tag{7.11}$$

there are low-energy states associated with slow motion in the vacuum manifold

$$< \bar{g}_{Li}(x), g_R^j(x) >= \nu \Sigma_i^j(x) \ (\Sigma \in SU(3)) \tag{7.12}$$

Write

$$\Sigma = exp(\frac{ZiM}{f}) \tag{7.13}$$

$$M = \begin{pmatrix} \frac{\pi^\circ}{\sqrt{2}} + \frac{n}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^\circ}{\sqrt{2}} + \frac{n}{\sqrt{6}} & K^\circ \\ K^- & \bar{K^\circ} & -\frac{2n}{\sqrt{6}} \end{pmatrix} \tag{7.14}$$

$$L = \frac{f_{\bar{u}}^2}{8} tr \partial^\mu \Sigma^+ \partial_\mu \Sigma \tag{7.15}$$

gives the properly normalized axial current.
Note

$$\begin{array}{cc} SU(3)_{L+R} & flavor\ symmetry \Sigma \to U^+ \Sigma U \\ SU(3)_L \times SU(3)_R & \Sigma \to U^+ \Sigma U \end{array} \tag{7.16}$$

No potential is allowed, since $\Sigma\Sigma^+ = 1, det\Sigma = 1$.
Quark masses likewise transform as $U^+ \mathcal{M} \mathcal{V}$ (since they go with $\bar{g}_{Li} g_R^j$).
So

$$\Delta L \equiv \upsilon Tr(m_q^+ \Sigma + m_q^+ \Sigma^+) + highness\ in\ \partial, m \tag{7.17}$$

with

$$\mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \tag{7.18}$$

$$\Delta L = -\frac{4\upsilon^2}{f^2} tr \begin{pmatrix} \frac{\pi^{\circ 2}}{2} + \frac{n^2}{6} + \pi^+\pi^- + K^+K^- + \frac{\pi^\circ n^\circ}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{\pi^{\circ 2}}{2} + \frac{n^2}{6} + \pi^+\pi^- + K^\circ \bar{K^\circ} + \frac{\pi^\circ n^\circ}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2}{3}n^2 + K^+K^- + K^\circ \bar{K^\circ} \end{pmatrix}$$

$$\times \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \tag{7.19}$$

$$m^2_{\pi^\circ\pi^\circ} = \frac{4v^2}{f^2}(m_u + m_d) \tag{7.20}$$

$$m^2_{\eta\eta} = \frac{4v^2}{f^2}(\frac{m_u + m_d}{3} + \frac{4m_s}{3}) \tag{7.21}$$

$$m^2_{\pi^\circ\eta} = \frac{4v^2}{f^2}\frac{m_u - m_d}{\sqrt{3}} \tag{7.22}$$

$$m^2_{K^+K^-} = \frac{4v^2}{f^2}(m_u + m_s) \tag{7.23}$$

$$m^2_{K^\circ\bar{K}^\circ} = \frac{4v^2}{f^2}(m_d + m_s) \tag{7.24}$$

Phenomenologically:

$$m_u, m_d \ll m_s \tag{7.25}$$

$m_u - m_d$ is not much smaller than $m_u + m_d$. It is still rather poorly determined. Gets mixed up with QED corrections.

Probably

$$\frac{m_u}{m_d} \approx 0.4 \tag{7.26}$$

$$\frac{m_u + m_d}{m_s} \approx \frac{1}{25} \tag{7.27}$$

$\pi - n$ mixing is $\sim \frac{m_u+m_d}{m_s}$.

From all this we get

$$3m^2_\eta + m^2_{\pi^\circ} = 2(m^2_{K^+} + m^2_{K^\circ}) \tag{7.28}$$

which works very well.

However, if we include a singlet

$$M = \begin{pmatrix} \sigma + \frac{n}{\sqrt{6}} & & \\ & \sigma + \frac{n}{\sqrt{6}} & \\ & & \sigma - \frac{2n}{\sqrt{6}} \end{pmatrix} \tag{7.29}$$

however normalized, $\sigma - \frac{2n}{\sqrt{6}}$ gets no contribution from $m_s \Rightarrow$ extra light pseudoscalar mesons Has not been seen (n' won't do). This is the $U_A(1)$ problem.

Chapter 8

Meson Mass Matrix

Close up on meson mass matrix states as:

$$\begin{aligned} V_m &= \upsilon\{(\frac{1}{F}\frac{a}{\sqrt{3}}+\frac{1}{8}\frac{\pi^\circ}{\sqrt{2}}+\frac{1}{f}\frac{n}{\sqrt{6}})^2 m_u + \\ &\quad (\frac{1}{F}\frac{a}{\sqrt{3}}-\frac{1}{8}\frac{\pi^\circ}{\sqrt{2}}+\frac{1}{f}\frac{n}{\sqrt{6}})^2 m_d + \\ &\quad (\frac{1}{F}\frac{a}{\sqrt{3}}-\frac{1}{f}\frac{2n}{\sqrt{6}})^2 m_s\} \qquad (8.1) \\ &= \frac{\upsilon}{2f^2}\begin{pmatrix} a & \pi^\circ & \eta \end{pmatrix} \\ &\quad \begin{pmatrix} \frac{f^2}{F^2}\frac{2}{3}(m_u+m_d+m_s) & \frac{2f}{F}\frac{m_u-m_d}{\sqrt{6}} & \frac{2f}{F}\frac{m_u+m_d-2m_s}{3\sqrt{2}} \\ \frac{2f}{F}\frac{m_u-m_d}{\sqrt{6}} & (m_u+m_d) & \frac{1}{\sqrt{3}}(m_u-m_d) \\ \frac{2f}{F}\frac{m_u+m_d-2m_s}{3\sqrt{2}} & \frac{1}{\sqrt{3}}(m_u-m_d) & \frac{1}{3}(m_u+m_d+4m_s) \end{pmatrix} \\ &\quad \begin{pmatrix} a \\ \pi^\circ \\ \eta \end{pmatrix} \qquad (8.2) \end{aligned}$$

So

$$\frac{\mathcal{M}^\in}{(\frac{\upsilon}{f^2})} = \begin{pmatrix} \frac{f^2}{F^2}\frac{2}{3}(m_u+m_d+m_s) & \frac{2f}{F}\frac{m_u-m_d}{\sqrt{6}} & \frac{2f}{F}\frac{m_u+m_d-2m_s}{3\sqrt{2}} \\ \frac{2f}{F}\frac{m_u-m_d}{\sqrt{6}} & (m_u+m_d) & \frac{1}{\sqrt{3}}(m_u-m_d) \\ \frac{2f}{F}\frac{m_u+m_d-2m_s}{3\sqrt{2}} & \frac{1}{\sqrt{3}}(m_u-m_d) & \frac{1}{3}(m_u+m_d+4m_s) \end{pmatrix} \qquad (8.3)$$

1. For $\frac{f}{F} \ll 1$ (axions). There is a small eigenvalue $\sim \frac{f^2}{F^2}$ associated with an eigenvalue $\sim \begin{pmatrix} 1 \\ \frac{f}{F}a \\ \frac{f}{F}b \end{pmatrix}$ with

$$2\frac{m_u - m_d}{\sqrt{6}} + (m_u + m_d)a + \frac{1}{\sqrt{3}}(m_u - m_d)b = 0 \quad (8.4)$$

$$\frac{\sqrt{2}}{3}(m_u + m_d - 2m_s) + \frac{1}{\sqrt{3}}(m_u - m_d)a + \frac{1}{3}(m_u + m_d + 4m_s)b = 0 \quad (8.5)$$

After some algebra we find

$$a = -\sqrt{\frac{3}{2}}\frac{m_s(m_u - m_d)}{m_u m_d + m_u m_s + m_d m_s} \quad (8.6)$$

$$b = \frac{1}{\sqrt{2}}\frac{m_u m_s + m_d m_s - 2m_u m_d}{m_u m_d + m_u m_s + m_d m_s} \quad (8.7)$$

and $(mass)^2$

$$\mu^2 = \frac{3m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s}\frac{\upsilon}{F^2} \quad (8.8)$$

$$\simeq \frac{3m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s}\frac{2}{m_u + m_d}\frac{f^2}{F^2}m_\pi^2 \quad (8.9)$$

2. In opposite limit $\frac{f}{F} \gg 1$, note that if we take $m_u = m_d = 0$

$$\frac{\mathcal{M}^{\in}}{(\frac{\upsilon}{f^2})} = \begin{pmatrix} \frac{f^2}{F^2}\frac{2}{3}m_s & 0 & \frac{f}{F}\frac{-4}{3\sqrt{2}}m_s \\ 0 & 0 & 0 \\ \frac{f}{F}\frac{-4}{3\sqrt{2}}m_s & 0 & \frac{4}{3}m_s \end{pmatrix} \quad (8.10)$$

has two vanishing eigenvalues. So η gets infected and the GM-O relation is badly violated. The general case is a little messy but with $m_u = m_d \ll m_s$ we easily arrive at

$$m^2_{infected\ \eta} \to \frac{\upsilon}{f^2}3\frac{(m_u + m_d)m_s}{m_u + m_d + m_s} = 3m_\pi^2 \quad (8.11)$$

In general, this serves as a bound for all values of $\frac{F}{f}$. The low light meson is unacceptable unless it is very weakly coupled. This is the $U_A(1)$ problem.

Chapter 9

Resolution of $U_A(1)$

Resolution of the $U_A(1)$ problem: We have apparently,

$$\partial_\mu j^{\mu 5} \approx \epsilon^{\alpha\beta\gamma\delta} G^a_{\alpha\beta} G^a_{\gamma\delta} (+mass\ terms) \tag{9.1}$$

so that symmetry is removed. However, for the Abelian case we saw that the right-hand side is a total derivative, and this continues to be true in the non-Abelian case. Indeed

$$\begin{aligned} \epsilon^{\alpha\beta\gamma\delta}(\partial_\alpha A^a_\beta + f^{abc} A^b_\alpha A^c_\beta)(\partial_\gamma A^a_\delta + f^{ade} A^d_\gamma A^e_\delta) &= \partial_\alpha(\epsilon^{\alpha\beta\gamma\delta}(A^a_\beta \partial_\gamma A^a_\delta + \frac{2}{3} f^{abc} A^a_\beta A^b_\gamma A^c_\delta)) \\ &\equiv \partial_\alpha K^\alpha \end{aligned} \tag{9.2}$$

So there is a modified current that generates a legitimate symmetry.

This is not necessary a disacter, because the current is not gauge-invariant. So the massless states it produces might not be physical (that's what happens in the Meissner-Higgs effect), but this is not convincing unless we have a mechanism.

Here is the basic story-line:

1. Since K^α is not gauge-invariant, it does not appear directly in the action, and it may fluctuate more wildly. This could prevent the surface terms in

$$\int \partial j - \int \epsilon GG = \int \partial K = \int_S K dS \tag{9.3}$$

   from vanishing.

2. Concretely, we can imagine $A \sim \uparrow_r^\infty \frac{1}{r}$ in such a way that $K \sim \frac{1}{r^3}$ but G^2 falls off faster.

3. In fact this will happen if A approaches a non-trivial (direction dependent) pure gauge configuration.

$$A_\alpha \to \Omega^{-1}\partial_\alpha\Omega \tag{9.4}$$

4. Specifically, the integral $\int K^\alpha dS^\alpha$ is proportional to the degree of the mapping from S^3, the boundary of (Euclideanized) space-time into the manifold of $SU(2)$, which is also topologically S^3.

   Mathematical aside: $SU(2)$ matrices: $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, |\alpha|^2+|\beta|^2 = 1 \Rightarrow S^3$. Other groups: deform map into a sphere $\pi^3(G) = Z$ for any group G.

5. From the inequality

$$\int tr(G_{\mu\nu} - \tilde{G}_{\mu\nu})^2 \geq 0 \tag{9.5}$$

   (with $\tilde{G}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$) we derive

$$\int trG_{\mu\nu}G_{\mu\nu} \geq |\int trG_{\mu\nu}\tilde{G}_{\mu\nu}| \quad (\geq 8\pi^2) \tag{9.6}$$

   Thus the weight $e^{-\frac{1}{g^2}\int GG}$ of configurations that violate the conservation law is $\leq e^{-\frac{8\pi^2}{g^2}}$. It vanishes to all nodes in perturbation theory.

   Scholium:

   As in the Higgs effect the massless would be Nambu-Goldstone modes are subject to long-range forces involving gauge fields, but here the relevant gauge fluctuations are topological and quantized. Because of this, the mass-generation mechanism is essentially nonperturbative. Of course, this had to be so in QCD, since the only mass scale is $\sim e^{-\frac{8\pi^2}{g^2}}$.

6. The bound is saturated when

$$G_{\mu\nu} = \pm\tilde{G}_{\mu\nu} \tag{9.7}$$

   This can be solved fairly simply for the minimal charges, and using very elaborate mathematics for higher charges. However, the precise solutions are limited value for practical QCD, basically, because the coupling is just uniformly small.

7. The detailed way in which $U_A(1)$ violation shows up – in terms of changes in fermion quantum numbers or occurance of otherwise forbidden processes, is quite interesting and important in other contexts.

   When calculating a transition amplitude using path integrals, one must calculate the fermion determinant. The integrated anomaly equation

$$\Delta Q_5 = (\Delta winding) \times (\#\ doublets) \tag{9.8}$$

   suggests that this determinant will vanish when $\Delta winding$ is evaluated between states in which the ΔQ_5 is not appropriate. This signals zero-energy solution, a zero-modes of the Dirac equation. There is a corresponding mathematical theorem, the Atiyah-Singer index theorem.

8. To evaluate processes in the nonvanishing sector, we must allow for creation and destruction of fermions. This is done by coupling in source and taking derivatives, e.g.,

$$\begin{aligned} < \bar{\psi}_{x_1}\bar{\psi}_{x_2}\psi_{x_3}\psi_{x_4} > \ \sim\ & \frac{\delta}{\delta\eta(x_1)}\frac{\delta}{\delta\eta(x_2)}\frac{\delta}{\delta\bar{\eta}(x_3)}\frac{\delta}{\delta\bar{\eta}(x_4)}|_{\eta=0} \\ & \times \frac{\int e^{-(S_{gauge}+\bar{\psi}\nabla\psi+\bar{\eta}\psi+\bar{\psi}\eta)}}{\int e^{-(S_{gauge}+\bar{\psi}\nabla\psi)}} \end{aligned} \tag{9.9}$$

   The zero-modes give $\nabla\psi_0 \sim \eta\psi_0$ and so directly a prove of η (or $\bar{\eta}$) in the determinant. We get a nonzero answer by solving them up with $\frac{\delta}{\delta\eta}$ factor. This enforces the anomaly equation (for more algebraic details see Coleman).

   Scholium:

   In the context of electroweak theory, the $SU(2)_L$ anomaly equation suggests a mechanism of baryon number violation. I have asked you to spell this out as a problem.

Chapter 10

Strong P, T Problem

One result of the analysis is that a term $\approx \int trG\tilde{G}$ in the action of QCD is not negligible, even though, it is a total derivative (so it does not appear in the classical equations of motion).

Since it is a dimension 4 interaction, it should appear in our canonical form for QCD. Unfortunately it violates P and T, so its processes with a substantial coefficient would not agree with experience. Detailed estimates suggest

$$|\frac{\theta}{2\pi}| \quad (the\ natural\ parameter) \leq 10^{-9} \tag{10.1}$$

Since there is P and T violation in the electroweak interaction, this term will be needed for renormalization so unless there is an extra symmetry to explain why $\theta \approx 0$ (or can be rotated away) we have an ugly situation.

The required symmetry Peccei-Quinn) is essentially axial baryon number, since this counts ($\Delta winding$) and thus powers of $e^{i\theta}$.

If there is a symmetry of this type and spontaneously broken at a high scale F, the kinetic term of its Nambu-Goldstone phase field reads

$$\partial_\mu \varphi \partial_\mu \varphi^+ \overset{\varphi = Fe^{i\theta}}{\longrightarrow} F^2(\partial_\mu \theta)^2 \tag{10.2}$$

This puts us back at our earlier analysis. We get, for large F a very light, very weakly interacting particle, the axion.

It is a serious candidate to supply the dark matter of the universe.

For more on axioms, see my ′83 Erice lectures.

Chapter 11

Lattice Gauge

As I have mentioned repeatedly, this is the ultimate definition of QCD. (For electroweak theory, there is no satisfactory non-perturbative definition). I also discussed before the process of dimensional transmutation, you should refer back to this after we have gone through the explicit construction of LGT.

Agenda here:

1. Formulation of pure gauge theory.
2. Formulation of fermion theory, doubling phenomenon.
3. Confinement in strong coupling

Euclideanize, introduces $4d$ cubic lattice. On links introduce (for QCD) $SU(3)$ matrices $\underbrace{U_{n_1,n_2}}_{site\ lables}$.

They should be thought of as parallel transporters, i.e., solution of the equation

$$\begin{aligned} \nabla_\mu U &= 0 && (11.1)\\ &= \partial_\mu U + igA_\mu U && (11.2)\\ U &= P\ (ordered\ integral) && (11.3) \end{aligned}$$

Thus, if, say,

$$\psi(x) = U(x, x_0)\psi_0 \tag{11.4}$$

then

$$\nabla_\mu \psi = 0 \tag{11.5}$$

Note that there is path-dependence in this definition of $\psi(x)$ and the parallelism is only along the path.

Let me emphesis, though, that this is only a mnemonic, for relating to the (formal) continuous theory. LGT itself does not know about A.

$$U_{n_1 n_2} = U^{-1}_{n_2 n_1} \tag{11.6}$$

We will enforce a gauge symmetry

$$U_{n_1 n_2} \to \Omega(n_1) U_{n_1 n_2} \Omega^{-1}(n_2) \tag{11.7}$$

with site-dependent $\Omega(n) \in SU(3)$. Thus, the symmetry group is $SU(3)^{Z^4}$

The simplest nontrivial local invariant is the trace around a plaquette

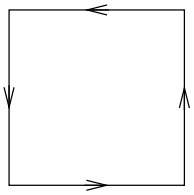

Figure 11.1: Plaquette.

$$trU_{n,n+\hat{x}} U_{n+\hat{x},n+\hat{x}+\hat{y}} U_{n+\hat{x}+\hat{y},n+\hat{y}} U_{n+\hat{y},n} \equiv tr\Box_{n,xy} \tag{11.8}$$

For small fluctuations this is ≈ 3

This inspires the action

$$S = \frac{c}{g^2} \sum_{plaquettes} (3 - tr\Box) \tag{11.9}$$

where, $c = \frac{1}{24}$ to match continuous conventions.

To be completely explicit we should also specify the measure. It is the product of Haar measures $\prod_{links}[dU]$

$$[dU] = \int U^{-1}dU \underbrace{\wedge \cdots \wedge}_{8\ times} U^{-1}dU \tag{11.10}$$

$$= \int \underbrace{dx_i}_{\substack{group\ manifold \\ parametrization\ of}} \underbrace{det}_{jacobian} \underbrace{||\frac{U^{-1}(x)dU(x)}{\partial x_i}}_{3\times 3\ traceless\ Hermitean} || - use\ normalized\ b(11.11)$$

Crucial property is

$$[d(U_0U)] = [dU] \tag{11.12}$$

thus

$$\int [dU](non - singlet) = 0 \tag{11.13}$$

e.g.

$$\begin{aligned} \int [dU]U &= \int [d(U_0U)]U && (11.14)\\ &= \int [d\tilde{U}]U_0^{-1}\tilde{U} && (11.15)\\ &= U_0^{-1}\int [dU]U && (11.16)\\ \int [dU]U &= 0 && (11.17) \end{aligned}$$

Note:

1. For small fluctuations, first non-trivial term is quadratic. Thus it must match continuous $\int trG_{\mu\nu}G_{\mu\nu}$
2. Everything is numerical – no units ($\rightarrow$ dimensional transmutation).
3. Everything is finite.
4. Everything is algorithmic.
5. No gauge fixing required.

Fermions (quarks) live on sites they transform as

$$\psi_n \rightarrow U\psi_n \tag{11.18}$$

The simplest kinetic energy is

$$\frac{1}{2i}\sum_{n,unit}\ \sum_{displacement,\delta} \bar{\psi}_{n+\delta}\gamma_\delta U_{n+\delta,n}\psi_n \tag{11.19}$$

(i.e., $\gamma_{\hat{x}} = \gamma_1, \gamma_{-\hat{x}} = -\gamma_1, \cdots$)
Consider $U \approx 1$, plane wave

$$\psi_n \sim e^{ip.n}S \tag{11.20}$$

$(p.n = \underbrace{p_1 n_1}_{integer} + p_2 n_2 + p_3 n_3 + p_4 n_4)$.

We have

$$\frac{1}{2i}\sum_n (\bar{\psi}_{n+\hat{x}} - \bar{\psi}_{n-\hat{x}})\gamma_{\hat{x}}\psi_n \to \sin p_x \gamma_{\hat{x}} \tag{11.21}$$

so inverse propagator $\gamma_i \sim p_i$ low energy states (= poles of propagator) for

$$\sum_i (\gamma_i \sin p_i)^2 = 0 \tag{11.22}$$

$$= \sum_i \sin^2 p_i \tag{11.23}$$

This occurs near $p_i \approx 0$ – smooth fields, but also for any $p_i = \pi$

Near $p_i = \pi$ the direction of E as p_i is reversed, so chirality is opposite.

This introduces 16 branches, where we wanted 1. You can scheme this effect but not eliminate it, by simple modifications. Recently more basic methods, involving adding considerable additional structure ($\approx$ extra dimensions, $4d$ domain wall) have emerged.

The brutal way (Wilson) is to add

$$-\frac{1}{2}\sum_{n,\delta} \bar{\psi}_{n+\delta} U \psi_n + \underbrace{4\sum \bar{\psi}_n \psi_n}_{4-\cos p_x - \cos p_y - \cos p_z - \cos p_\tau} \tag{11.24}$$

This eliminates the small energy at any $\cos p = -1$. It also, of course, violates chiral symmetry explicitly. The 4 must actually, be tuned, in an interaction dependent way, to get a light branch.

Massive quarks will stay put, so adding a massive quark-antiquark pair separated at distance R for time T inserts U matrices.

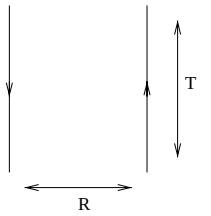

Figure 11.2: RT.

To determine the potential therefore we evaluate

$$e^{-V(R)T} = \lim_{T\to\infty} \frac{tr \int [dU] \int e^{-\frac{c}{g^2}\sum_{plaquette}\Box}}{\int [dU] \int e^{-\frac{f}{g^2}\sum \Box}} \tag{11.25}$$

In story coupling, expand the action

$$e^{-\frac{c}{g^2}\sum D} = \prod(1 - \frac{c}{g^2}\sum \square \cdots)) \tag{11.26}$$

The 1 term works fine in the denominator, but the integration over links appearing in the large loop will vanish. To get a non-zero answer we must pair these links.

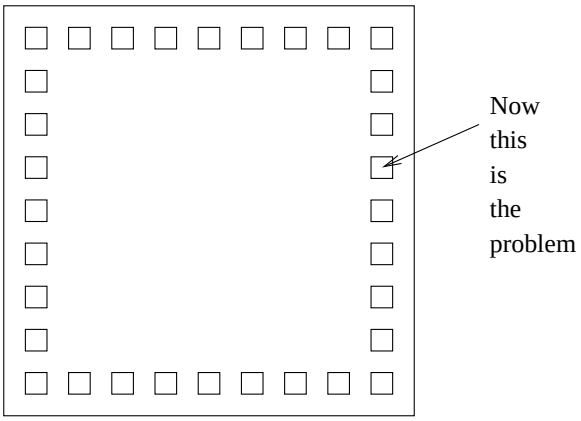

Figure 11.3: Sheet.

Keep going until the whole sheet is filled in. This gives a factor

$$e^{-V(R)T} \sim (\frac{1}{g^2})^{RT} \tag{11.27}$$

so $V(R) \propto R$.

Linear potential ($\Rightarrow$ confinement) is manifest at strong coupling. Of course, the continuous theory – fine lattice spacing – corresponds to weak coupling, as we saw earlier. If there is no phase transition, we get confinement there too. This has been shown numerically for QCD. QED ($U(1)$), on the other hand, has a phase transition.